Current Induced Modification of the Ferromagnetic Magnon Dispersion Relation

by

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Dedicated to Scrolan.

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Current Induced Modification of the Ferromagnetic Magnon Dispersion Relation

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In this masters thesis I discuss the influence of a spin polarized current on the magnetization of a ferromagnetic metal from a microscopic point of view.

The starting point is the Hubbard Model, solved with the coherent path integral formalism. To include the current in the calculation, the partition sum is restricted to only those fermionic states which carry a current. The magnons are derived as Gaussian fluctuations around the classical state.

The results of the calculation are several modifications of the response function. The changed dissipation may be able to explain some measurements by Myers et al. [1] and Tsoi et al. [2]. The change in the dispersion relation reproduces the result of Bazaliy, Jones and Zhang [3], but refute their assumption of a relayted instability. Instead the magnons are translated in a frame of reference, which is moving with the current flow.

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Chapter 1

Introduction

In the late eighties, various magneto-resistance effects [4] were discovered. In magneto-resistance devices, the magnetization exerts an influence on the current going through it. But according to Netwon's 3^{rd}law, every force has its counterpart. So if the magnetization acts on the current, should not the magnetization be influenced by the current?

This is not only a principal question, it is also interesting, for technological applications.

1.1 Writing a MRAM Bit

J. Slonczewski [5] proposed an effect which enables to write an MRAM Bit with only a current going through. Compared to writing with a magnetic field, writing with current provides the possibility to write much more dense structures. A current can be much better focused than a magnetic field, so the influence on the neighboring bits is much smaller, which is one of the limiting factors of todays data storage industries.

A MRAM bit consists of a FM/M/FM multilayer, one ferromagnetic layer is fixed and one is switchable. A small metallic spacer, which is much smaller than the spin diffusion length decouples the two ferro-magnets. Due to the giant magneto-resistance, a resistivity measurement can determine the anti-parallel or the parallel configuration, of the magnetization directions, which resembles a 0 or 1.

The first ferromagnetic layer polarizes the current $\vec{I}$ going through parallel to its magnetization direction $\vec{M}_1$. The so polarized current enters the second switchable ferromagnetic layer, which is polarized in direction $\vec{M}_2$. If the magnetization directions are not (anti)parallel, then the polarized electrons start to precess around $\vec{M}_2$.

Due to the dispersion relation and scattering processes the electrons loose their relative phase, and the average magnetization in the x-y plane decays, while the magnetization in z-direction is preserved.

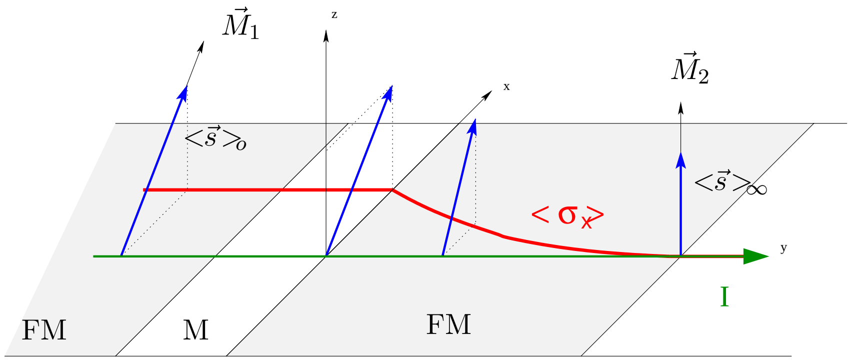


Figure 1.1: Slonczevski's Approach of MRAM Bit Writing

Due to angular momentum conservation as Slonczewski argues, the magnetization has to absorb this angular momentum and tilt. This represents a force acting on the magnetization, which is described by the altered Landau Lifschitz equation:

$$\frac{\partial \vec{n}_1}{\partial t} = \rho \Delta \vec{n}_1 \times \vec{n}_1 - \frac{g\, I}{e} (\vec{n}_1 \times \vec{n}_2) \times \vec{n}_1 \tag{1.1}$$

With the current I, and a numerical factor g, which contains the amount of polarization and current going through the system. At a certain amount of current, this force overcomes the anisotropy and the bit switches. The most exciting point of this writing mechanism is, it can be reversed by reversing the direction of current flow.

The effect verified by E.B.Myers et al. [1], with a Co-Cu-Co sandwich structure, next to a nanoconstriction. Nano structuring techniques are necessary to attain the required current densities of up to $10^9\ A/cm^2$.

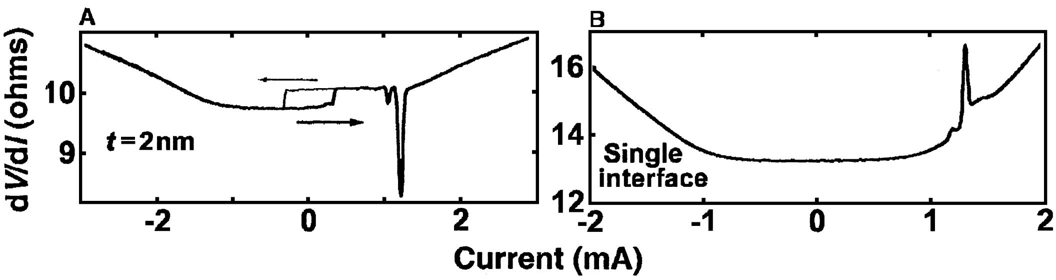


Figure 1.2: Current induced magnetization switching

The figure 1.1A shows a small hysteresis like structure. This is caused by current induced flipping of the magnetization, and the resulting change in the giant magneto-resistance.

But there is another feature at higher currents. At approximately a current of +1.2mA, there is a step in the resistivity. This happens not only in the sandwich structure, but also at a nanoconstriction connecting Cu and Co, as in figure 1.1B. This step in resistivity is referred to an excitation of the multi layers.

1.2 Spin Wave Laser

This excitation of the layers was theoretically predicted by Berger [6], and experimentally discovered as well by Tsoi [2] at a point contact on Co-Cu multi layers.

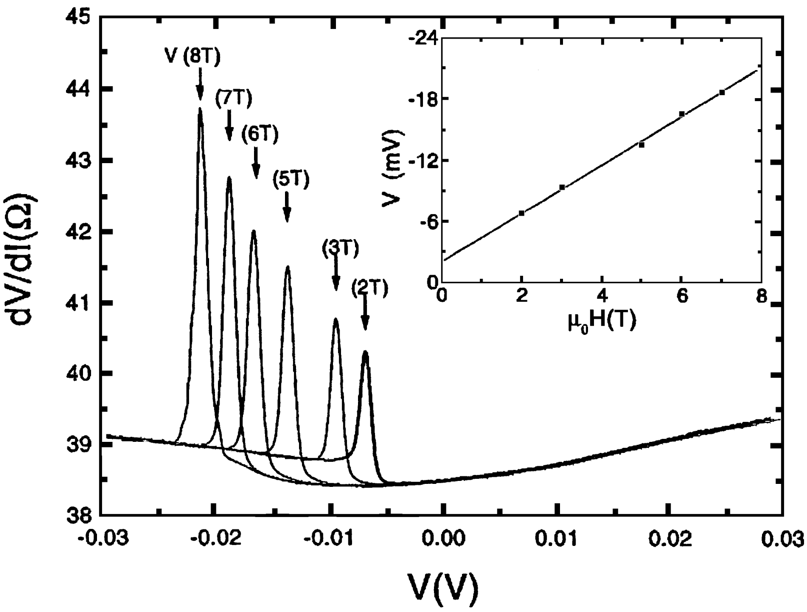


Figure 1.3: Current induced rise of resistivity

In a more recent paper [7] Tsoi exposes the point contact to microwave radiation of 50.6 GHz, which pumps eventually a spin wave maser (SWASER), another prediction of Berger [8].

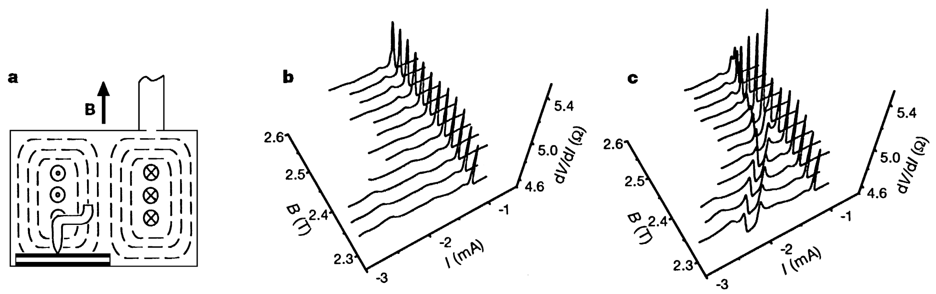


Figure 1.4: Additional structure, possible Swaser

Figure 1.3a shows the experimental setup. Figure 1.3b shows measurements without and figure 1.3c with radiation. As you can see, on the nearly stationary peak, this happens at currents, which lie beyond the current necessary to produce the resistivity step.

1.3 Theoretical approaches

Many aspects of the theoretical description of these effects remain obscure, partly because a first quantisation electron wave functions scattering at rectangular potentials with a phenomenological coupling to the magnetization is used as a describing language.

A different approach was made by Bazaliy, Jones and Zhang [3]. They solved the Schroedinger equation of an electron in a special magnetic field.

This field rotates from a fixed direction at the origin to the easy axis inside the bulk. This model shall describe the relaxation of a spin polarized current, which fixes the magnetization at the origin. Through this model they derived an extra term in the Landau-Lifschitz equation.

$$\frac{\partial \vec{n}}{\partial t} = \left(\widetilde{J} \Delta \vec{n} + \widetilde{K} (\vec{\nu} \vec{n}) \vec{n} \right) - \frac{Q}{e} j_i \nabla_i \vec{n} \tag{1.2}$$

They showed that this new term cause a spin wave instability at high enough currents. But they introduce the current j as a factor, to adjust the units. So the relation of j and the physical spin polarized current is not stringent.

To improve the understanding we tried to model these experiments from a microscopic point of view in a modern many body formalism.

From the numerous microscopic models available to describe a ferromagnet, we choose the Hubbard model in the path integral formalism, as pointed out by R. E. Prange and V. Korenman [9]. This method, combines two paradigms, the collective spin wave modes easiest calculated from the Heisenberg model and the band splitting due to the Zeeman energy.

The path integral approach has also another advantage. It provides the possibility to include a current in a non perturbative way. By restricting the summation of the partition sum to only current carrying states, the current is included without changing the model Hamiltonian.

This approach is elaborated in the following thesis.

Chapter 2

Technical Introduction I: Spin Wave Theory

The main objective of this project is to discuss the effects of a current on spin waves. Before starting with them, I derive the current free solution of the Hubbard Model. I will refer later to this, indicating the steps of the calculation at which changes occur when a current flows. This Chapter also introduces to my notation. A useful reference for this topic is Fradkin [10].

2.1 The Hubbard Hamiltonian

In the tight binding model the kinetic energy is described as a hopping matrix t_{ij} which enables the fermions to travel from the lattice site i to j. The Pauli exclusion principle is enforced by the anti-commutation relation of the creation and annihilation operators $c^\dagger$ and c.

To include the electrostatic interaction of the fermions, Hubbard [11] [12] [13] suggested a crude simplified additional term. The interaction is reduced to an energy cost U to put two fermions on the same lattice site. The so called Hubbard Hamiltonian is

$$H = \sum_{i,j,\sigma} t_{ij} c^\dagger_{i\sigma} c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{2.1}$$

Even with this crude approximation this model contains very interesting physics. Unfortunately it can only be be solved exactly in the one dimensional case [14].

A convenient approach, to describe ferromagnetism in this model is the auxiliary field method. To do this, we first have to rewrite the interaction term as the square of the spin operator at each lattice site i. The simplest way to establish the identity we need is to look at all four possible occupation possibilities of a lattice site.

Table 2.1: Mapping of $n_{i\uparrow} \cdot n_{i\downarrow}$ up to $\vec{S}_i^2$

site i	$n_{i\uparrow} n_{i\downarrow}$	$n_{i\uparrow} + n_{i\downarrow}$	S_i^2
$- -$	0	0	0
$\uparrow -$	0	1	3/4
$- \downarrow$	0	1	3/4
$\uparrow\downarrow$	1	2	0

So we can rewrite the Hamiltonian as

$$H = \sum_{i,j,\sigma} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + \frac{U}{2} \sum_{i} \left(n_{i\uparrow} + n_{i\downarrow}\right) - \frac{2U}{3} \sum_{i} S_i^2 \tag{2.2}$$

Because the sum of the spin up and down occupation operators counts every electron once, boths can be absorbed in the chemical potential of the system. Therefore we shift the chemical by $U/2$, which is contained in the hopping matrix t_{ij} as diagonal entries and write:

$$H = \sum_{i,j,\sigma} t'_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} - \frac{2U}{3} \sum_{i} S_i^2 \tag{2.3}$$

Finally the spin operator can be expressed in terms of creation and annihilation operators.

$$H = \sum_{\vec{r}\vec{r}'\sigma} c^{\dagger}_{\vec{r}\sigma}\, t'_{\vec{r}\vec{r}'}\, c_{\vec{r}'\sigma} - \frac{1}{6}U \sum_{\vec{r}\sigma\sigma'} \left[c^{\dagger}_{\vec{r}\sigma} \vec{\tau}_{\sigma\sigma'} c_{\vec{r}\sigma'} \right]^2 \tag{2.4}$$

With this Hamiltonian, the grand canonical partition function can be calculated.

2.2 Derivation of the Effective Action

The partition function is introduced with the same notation and conventions as in the textbook of Negele & Orland [15]

$$Z = \int D\bar{\psi} D\psi \exp[-S] \tag{2.5}$$

The coherent states are eigenstates of the annihilation operator, so in the action the operators can be replaced by their Grassman eigenvalues ψ and $\bar{\psi}$:

$$S = \int_0^\beta d\tau \sum_{\vec{r}\vec{r}'\sigma} \bar{\psi}_{(\vec{r},\sigma)} \left[\partial_\tau - \mu + t_{\vec{r}\vec{r}'} \right] \psi_{(\vec{r}',\sigma)} - \frac{1}{6}U \sum_{\vec{r}\sigma\sigma'} \left[\bar{\psi}_{(\vec{r}\sigma)} \vec{\tau}_{\sigma\sigma'} \psi_{(\vec{r}\sigma')} \right]^2 \tag{2.6}$$

This integral can not be evaluated exactly, because the term $\vec{S}^2$ is quartic in the fermionic fields. But we know how to handle Gaussian integrals of Grassman variables, and Hubbard [16] and Stratonovich [17] developed a technique, that takes advantage of these identities and provides an alternative physical picture.

2.2.1 Hubbard Stratonovich Identity

The idea of the transformation is to introduce another Gaussian integral over an auxiliary field $\vec{h}_{(\tau_o,r_o)}$ plus one of the spin operators. Because the

Gaussian integral gives just an unimportant factor, we obtain an expression for $\vec{S}^2$, by expanding the exponent.

$$\exp\left[\lambda^2 \frac{\vec{S}^2_{(\tau_o\vec{r}_o)}}{2}\right] = \int d\vec{h}_{(\tau_o\vec{r}_o)} \exp\left[-\frac{\vec{h}^2_{(\tau_o\vec{r}_o)}}{2} + \lambda\vec{h}_{(\tau_o\vec{r}_o)}\vec{S}_{(\tau_o\vec{r}_o)}\right] \tag{2.7}$$

By introducing the bosonic field $\vec{h}$ at every point in space and time, the quartic fermion term can be removed. The price to pay is another path integral over the new bosonic field, so the partition function transforms into

$$Z = \int D\bar{\psi}D\psi \int D\vec{h} \exp\left[-S_{(\vec{h},\psi,\bar{\psi})}\right] \tag{2.8}$$

with the modified action

$$S_{(\vec{h},\psi,\bar{\psi})} = \int_0^\beta d\tau \sum_{\vec{r}} \frac{\vec{h}^2_{(\tau\vec{r})}}{2} + \sum_{\vec{r}\vec{r}'\sigma\sigma'} \bar{\psi}_{(\vec{r}\sigma)} \left[\partial_\tau - \mu + t_{\vec{r}\vec{r}'} - \sqrt{\frac{U}{3}}\vec{h}_{(\tau\vec{r})}\vec{\tau}_{\sigma\sigma'}\right] \psi_{(\vec{r}'\sigma')} \tag{2.9}$$

Because the bosonic field couples to the spins of the fermions we can interpret it as an effective magnetic field, that splits the energies of electrons with oposite spins.

The fermions are now only quadratic and can be integrated out exactly. But first let us Fourier transform the expression.

2.2.2 Fourier Transformation

In k-space, the hopping matrix t_{ij} becomes the kinetic energy ϵ_k which is diagonal in the momentum coordinate, which simplifies the problem. To save space, a covariant 4 momentum notation is used:

$$x = (\tau, \vec{r}) \quad k = (\omega_n, \vec{k}) \quad p = (\Omega_n, \vec{p}) \quad p \cdot x = -\omega_n\tau + \vec{p}\vec{x} \tag{2.10}$$

Here the capital Greek Ω_n stands for a bosonic and the lowercase Greek ω_n for a fermionic Matsubara frequency.

We define the Fourier transformations:

$$F(x) = \frac{1}{N\beta}\sum_k F(k)e^{ikx} \quad \text{and} \quad F(k) = \int_0^\beta d\tau \sum_{\vec{r}} F(x)e^{-ikx} \tag{2.11}$$

The necessary normalization constant N is the number of all lattice sites and β the imaginary time interval.

The Action becomes:

$$\begin{aligned} S &= \frac{1}{N\beta}\sum_q \frac{\vec{h}_{(q)}\vec{h}_{(-q)}}{2} \\ &+ \frac{1}{N\beta}\sum_{kq\sigma\sigma'} \bar{\psi}_{(k,\sigma)}\left[(-i\omega_n+\epsilon_k-\mu)\delta_{q,0}\,\delta_{\sigma,\sigma'} - \sqrt{\frac{U}{3}}\vec{h}_q\vec{\tau}_{\sigma\sigma'}\right]\psi_{(k+q,\sigma')} \end{aligned} \tag{2.12}$$

In the k-representation the magnetic field $h_{(x)}$ becomes non diagonal. I used the notation $h_{(q)}$ for a function of q and $h_{-q} \equiv h_{k,k+q}$ for an operator connecting the states $\langle k|$ and $|k+q\rangle$. It is worth noting the frequency coordinate of q is the difference of two fermionic frequencies, so it is a bosonic frequency $q^0 = \Omega_n$

Now we can integrate out the fermionic fields exactly and the result is an *effective action* for the auxiliary magnetic field

$$Z = \int Dh_{(q)} \exp[-S_{\text{eff}}] \qquad \text{with} \qquad S_{\text{eff}} = \frac{1}{N\beta}\sum_q \frac{|\vec{h}_{(q)}|^2}{2} - \mathrm{Tr}\ln[G^{-1}] \tag{2.13}$$

with the Green's function

$$G = \frac{1}{-i\omega_n + \epsilon_k - \mu - \sqrt{\frac{U}{3}}\vec{h}_q\vec{\tau}} = \frac{-1}{i\omega_n - (\epsilon_k - \mu - \sqrt{\frac{U}{3}}\vec{h}_q\vec{\tau})} \tag{2.14}$$

2.3 Mean Field Solution

Equation 2.13 is a quantum field theory for $\vec{h}$. The mean field solution, which resembles the classical solution, extremises the action. Thus it can be determined by setting the functional derivative of the action equal zero.

$$\begin{aligned} 0 = \frac{\partial S_{\text{eff}}}{\partial h^a_{(q_o)}} &= h^a_{(q_o)} - \frac{\partial}{\partial h^a_{(q_o)}} \operatorname{Tr} \ln[G^{-1}] \\ &= h^a_{(q_o)} - \frac{1}{(N\beta)^2} \sum_{kk',\sigma\sigma'} \langle k\sigma|G|k'\sigma'\rangle\langle k'\sigma'|\frac{\partial G}{\partial h^a_{(q_o)}}|k\sigma\rangle \end{aligned} \tag{2.15}$$

With the explicit functional derivative of the Green's function (2.14) and one $1/N\beta$ removed by the delta functions created through the functional derivation this equation reduces to

$$h^a_{(q_o)} = -\sqrt{\frac{U}{3}} \frac{1}{N\beta} \sum_{k\sigma} \langle k,\sigma|\, G\, \tau^a \,|k+q_o,\sigma\rangle \tag{2.16}$$

2.3.1 Guessing homogeneous Field

Actually, we know phenomenologically that we need a solution uniform in space and time if we want to describe a ferromagnet. This a-priori assumption turns out to be correct, as indicated by the positivity of the collective modes. If we have not chosen the right ground state, the spin wave dispersion relation would have shown an instability.

This choice has also a mathematical motivation. Due to the choice of a homogeneous field , the magnetic field operator $h_q = h_0$ becomes diagonal, ant the trace 2.16 can be solved exactly. This trace consists of a spin trace, a

Matsubara frequency trace and a momentum trace, which we discuss in turn below.

2.3.2 Spin Trace

The Green's function G is a 2x2 matrix in spin space. To perform the spin trace we have to separate the Green's function into a diagonal part and a sum of Pauli matrices, as described in Landau and Lifschitz [18].

$$\begin{aligned} \boldsymbol{G} = \frac{1}{X - \sqrt{\frac{U}{3}}\vec{h}_o\vec{\tau}} &= \frac{1}{2}\Big(\frac{1}{X - \sqrt{\frac{U}{3}}h_o} + \frac{1}{X + \sqrt{\frac{U}{3}}h_o}\Big)\cdot\mathbf{1} \\ &+ \frac{1}{2}\Big(\frac{1}{X - \sqrt{\frac{U}{3}}h_o} - \frac{1}{X + \sqrt{\frac{U}{3}}h_o}\Big)\frac{\vec{h}_o}{h_o}\cdot\vec{\boldsymbol{\tau}} \end{aligned} \tag{2.17}$$

To shorten the expressions we introduce the definitions

$$\begin{aligned} \frac{\vec{h}}{h} &= \vec{n} \\ \sqrt{\frac{U}{3}}h_o &= \Delta/2 \quad > 0 \\ E_k^{\pm} &= \epsilon_k - \mu \pm \Delta/2 \\ g_{\pm} &= \frac{1}{i\omega_n - (\epsilon_k - \mu \pm \Delta/2)} = \frac{1}{i\omega_n - E_k^{\pm}} \\ g_1 &= \frac{1}{2}(g_- + g_+) \quad \text{and} \quad g_\tau = \frac{1}{2}(g_- - g_+) \end{aligned} \tag{2.18}$$

The Green's function becomes

$$\begin{aligned} -G = \frac{1}{2}\Big(\frac{1}{i\omega_n - E_k^-} + \frac{1}{i\omega_n - E_k^+}\Big) &+ \frac{1}{2}\Big(\frac{1}{i\omega_n - E_k^-} - \frac{1}{i\omega_n - E_k^+}\Big)\vec{n}\vec{\tau} \\ \equiv \frac{1}{2}\Big(g_- + g_+\Big) &+ \frac{1}{2}\Big(g_- - g_+\Big)\vec{n}\vec{\tau} \\ \equiv g_1 &+ g_\tau\vec{n}\vec{\tau} \end{aligned} \tag{2.19}$$

where g_+ and g_- are just free particle Green's functions for spin up/down. Due to the magnetization, the bands are split by the energy Δ, which reflects the spontaneous Zeeman splitting. This is an important result and will be needed several times later on.

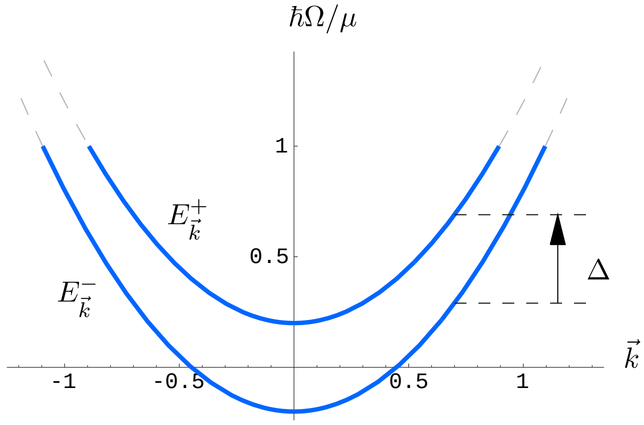


Figure 2.1: Zeeman splitting of the spin bands

Now the spin trace in (2.16) can be evaluated. As proved in Appendix A, the trace of one Pauli matrix vanishes. Thus the diagonal part of the Green's function g_1 will not contribute. Therefore the result is the difference of free Green's function like terms, which becomes Fermi Dirac distributions due to the Matsubara frequency sum:

$$\begin{aligned} h^a &= -\sqrt{\frac{U}{3}}\mathrm{Tr}[G\tau^a] \\ &= +\sqrt{\frac{U}{3}}\frac{1}{N\beta}\sum_{i\omega_n\vec{k}}(g_- - g_+)\, n^b\delta^{ba} \\ &= +\sqrt{\frac{U}{3}}\frac{1}{N}\sum_{\vec{k}}\left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})\right]n^a \end{aligned} \tag{2.20}$$

The minus sign cancels with the minus sign from the Green's function.

As a remark, the direction of the magnetization is arbitrary, and the $1/N$ factor vanishes, if we Fourier transform the expression back to real space. The resulting magnetic field is proportional to the total number of spin up electrons minus the number of down electrons.

2.4 Quadratic Fluctuations around Mean Field Solution

To study the dynamics of this spin system, quantum fluctuations around the homogeneous ground state magnetization are introduced.

$$\vec{h}_{(\vec{r},t)} = \vec{h}_o + \delta\vec{h}_{(\vec{r},t)} \quad \text{or} \quad \vec{h}_{(q)} = \vec{h}_o \cdot \delta_{(q)} + \delta\vec{h}_{(q)} \tag{2.21}$$

2.4.1 Effective Action with Fluctuations

This fluctuation in the field alters the field energy and also the Green's function. Because the inverse of the Green's function dependens only linearly on the field $\vec{h}$, we can separate it in the unperturbated inverse Green's function an a perturbation

$$G^{-1} = G_o^{-1} + \delta G_q^{-1} \qquad \text{with} \qquad \delta G_q^{-1} = -\sqrt{\frac{U}{3}}\,\vec{\tau}\,\delta\vec{h}_q \tag{2.22}$$

The action becomes:

$$S_{(\vec{h},\delta\vec{h})} \;=\; \frac{1}{N\beta}\sum_q \frac{(\vec{h}\cdot\delta_{(q)} + \delta\vec{h}_{(q)})^2}{2} - \operatorname{Tr}\ln[G_o^{-1} + \delta G_q^{-1}] \tag{2.23}$$

The perturbation Green's function G_q^{-1} is not diagonal, so the trace can not be solved exactly. But we can extract the original trace, and handle the rest though perturbation theory. Due to the logarithm property

$$\mathrm{Tr}\ln[G^{-1}] = \mathrm{Tr}\ln[G_o^{-1} + \delta G^{-1}] = \mathrm{Tr}\ln[G_o^{-1}] + \mathrm{Tr}\ln[1 + G_o\delta G^{-1}] \quad (2.24)$$

the action splits into the original unperturbated action and an action for the perturbating field δh.

$$S_{(\vec{h},\delta\vec{h})} = S_{o(\vec{h}_o)} + \frac{1}{N\beta}\vec{h}\ \delta\vec{h}_{(q=0)} + \frac{1}{N\beta}\sum_q \frac{\delta\vec{h}^2_{(q)}}{2} - \mathrm{Tr}\ln[1 + G_o\delta G^{-1}] \quad (2.25)$$

Assuming the perturbation is small, the trace can be expanded in a power series of $G_o\delta G^{-1}$.

$$\mathrm{Tr}\ln[1 + G_o\delta G^{-1}] \approx \mathrm{Tr}[G_o\delta G^{-1}] - \frac{1}{2}\mathrm{Tr}[G_o\delta G^{-1}G_o\delta G^{-1}] + O[\delta h^3] \quad (2.26)$$

The zeroth order is the unperturbated mean field solution. The first order should vanish, because h_o is the mean field solution. This property is verified explicit below. The second order is the part we are interested in. It will give us the quadratic fluctuations around the classical ground state, that we seek. All higher order contributions are neglected.

$$\begin{aligned} S_{(\vec{h},\delta\vec{h})} &= S_{o(\vec{h})} \\ &+ \frac{1}{N\beta}\vec{h}\delta\vec{h}_{(q=0)} - \mathrm{Tr}[G_o\delta G^{-1}] \\ &+ \frac{1}{N\beta}\sum_q \frac{\delta\vec{h}_{(q)}\,\delta\vec{h}_{(-q)}}{2} + \frac{1}{2}\mathrm{Tr}[G_o\delta G^{-1}G_o\delta G^{-1}] \end{aligned} \quad (2.27)$$

2.4.2 Vanishing 1. Order

To check if the linear contribution vanishes, we examine the trace linear in δh. Because G_o is diagonal in k, only the q=0 component of $\delta\vec{h}$ survives the sum over the momentum variable q.

$$\mathrm{Tr}[G_o\delta G^{-1}] = -\sqrt{\frac{U}{3}}\mathrm{Tr}[G_o\,\vec{\tau}\,\delta\vec{h}] = -\sqrt{\frac{U}{3}}\frac{1}{N\beta}\mathrm{Tr}[G_o\,\vec{\tau}]\delta\vec{h}_{(q=o)} \tag{2.28}$$

which is exactly the same as the altered field energy if $\vec{h}_o$ is replaced by its mean field expression. (2.20)

$$\frac{1}{N\beta}\vec{h}_o\delta\vec{h}_{(q=o)} = \frac{1}{N\beta}\sqrt{\frac{U}{3}}\mathrm{Tr}[G_o\vec{\tau}]\delta\vec{h}_{(q=o)} \tag{2.29}$$

As expected, the first order term vanishes.

2.5 Calculating the Bubble Diagram

The remaining effective action is

$$S_{(\vec{h},\delta\vec{h})} \quad = \quad S_{o(\vec{h})} + \frac{1}{N\beta}\sum_q \frac{\delta\vec{h}_{(q)}\,\delta\vec{h}_{(-q)}}{2} + \frac{1}{2}\mathrm{Tr}[G_o\delta G^{-1}G_o\delta G^{-1}] \tag{2.30}$$

To solve the trace of the Green's functions carefully, we write the trace over the momentums as a sum, and make all the 4-momentum vector indices explicit.

$$\mathrm{Tr}[...] = \frac{U/3}{(N\beta)^2}\sum_{kq}\mathrm{Tr}_\sigma[\,G_k\ \tau^a\delta h^a_{k,k+q}\ G_{k+q}\ \tau^b\delta h^b_{k+q,k}\,] \tag{2.31}$$

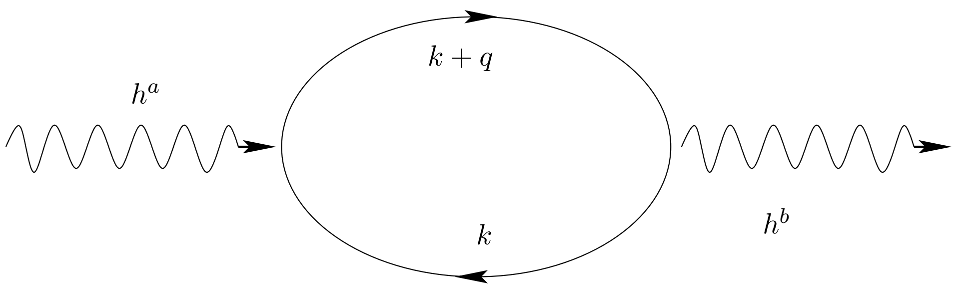
This trace corresponds to the graphical object called frequency Bubble Diagram: To calculate this diagram, we start with the spin trace.

2.5.1 Spin Trace

To solve the spin trace we separate the Green's function in its diagonal part, and parts proportional to the Pauli matrices ,just like in equation (2.19).

$$\mathrm{Tr}[...] = \frac{U}{3(N\beta)^2}\mathrm{Tr}\big[(g_{1(k)} + g_{\tau(k)}\vec{n}\vec{\tau})\delta\vec{h}_{(-q)}\vec{\tau}(g_{1(k+q)} + g_{\tau(k+q)}\vec{n}\vec{\tau})\delta\vec{h}_{(q)}\vec{\tau}\big] \tag{2.32}$$

Figure 2.2: Bubble Diagram in k-space



Now all the spin dependence of the terms are described by the Pauli matrices. As proved in Appendix A, the result of the spin trace is:

$$\begin{aligned} \mathrm{Tr}[...] = \frac{2U}{3(N\beta)^2} \sum_{kp} \quad & g_{1(k)} g_{1(k+q)} \cdot \delta\vec{h}_{(-q)} \delta\vec{h}_{(q)} \\ & + g_{\tau(k)} g_{\tau(k+q)} \cdot \left(2\vec{n}\delta\vec{h}_{(-q)}\, \vec{n}\delta\vec{h}_{(q)} - \delta\vec{h}_{(-q)}\delta\vec{h}_{(q)} \right) \\ + \left(g_{1(k)} g_{\tau(k+q)} - g_{\tau(k)} g_{1(k+q)} \right) & \cdot i\delta\vec{h}_{(-q)} \left(\vec{n} \times \delta\vec{h}_{(q)} \right) \end{aligned} \tag{2.33}$$

2.5.2 Separating the Polarizations

To improve the clarity of the result, we separate the fluctuation $\delta\vec{h}$ into the fluctuation $\vec{\sigma}$ parallel to the magnetization direction $\vec{n}$, and the transversal part $\vec{\pi}$.

$$S = S^o + \frac{1}{2N\beta} \sum_q K_\sigma \vec{\sigma}_{(-q)} \vec{\sigma}_{(q)} + K_\pi \vec{\pi}_{(-q)} \vec{\pi}_{(q)} + iK_{berry} \vec{\pi}_{(-q)} \left(\vec{n} \times \vec{\pi}_{(q)} \right) \tag{2.34}$$

The longitudinal kernel K_σ, the transversal kernel K_π and the Berry phase (see [19]) Kernel K_{berry} are given by the following expressions:

$$\begin{aligned} K_\sigma &= 1 + \frac{2U}{3N\beta}\sum_k g_{1(k)}g_{1(k-q)} + g_{\tau(k)}g_{\tau(k-q)} && (2.35)\\ K_\pi &= 1 + \frac{2U}{3N\beta}\sum_k g_{1(k)}g_{1(k+q)} - g_{\tau(k)}g_{\tau(k+q)} && (2.36)\\ K_{berry} &= \frac{2U}{3N\beta}\sum_k g_{1(k)}g_{\tau(k+q)} - g_{\tau(k)}g_{1(k+q)} && (2.37) \end{aligned}$$

2.5.2.1 Kernel of the longitudinal fluctuations $\vec{\sigma}$

We have split the problem in several smaller ones. Now we have to calculate each kernel. A helpful reference is the textbook by Negele & Orland [15]. First g_1 and g_τ are replaced by $g_\pm$ corresponding to (2.18).

$$\begin{aligned} K_\sigma &= 1 + \frac{2U}{3N\beta}\sum_k g_{1(k)}g_{1(k+q)} + g_{\tau(k)}g_{\tau(k+q)} \\ &= 1 + \frac{U}{6N\beta}\sum_k \big(g_{-(k)}g_{-(k+q)} + g_{+(k)}g_{+(k+q)}\big) && (2.38) \end{aligned}$$

Because g_+ and g_- have the free particle Green's function form, the result for the $k^o = i\omega_n$ Matsubara frequency sum is a Lindhard function.

$$\frac{1}{\beta}\sum_{i\omega_n\vec{k}} g_{\eta(k)}g_{\eta'(k+q)} = \sum_{\vec{k}} \frac{n_F(E^\eta_{\vec{k}}) - n_F(E^{\eta'}_{\vec{k}+\vec{q}})}{E^\eta_{\vec{k}} - E^{\eta'}_{\vec{k}+\vec{q}} + i\Omega_n} \qquad (2.39)$$

with the particle energies $E^\pm_{\vec{k}} = \epsilon_k - \mu \pm \Delta/2$. To obtain the retarded response function, we have to analytical continuate the imaginary frequency $i\Omega_n$ to the real axis, from the upper half plane: $i\Omega_n \to \Omega + i0^+$. To handle this $i0^+$, we need the equation

$$\int dx \frac{1}{f(x) \pm i0^+} = P\!\!\int dx \frac{1}{f(x)} \mp i\pi\delta_{(f(x))} \tag{2.40}$$

where the P indicates the Cauchy principle value of the integral. As you will see this polarization is not so interesting, so we neglect the imaginary part here.

Because we are interested in low energy physics, we are focused at the lowest order of Ω and $\vec{q}$. Therefore we start to examine the limit $\Omega = 0$ and $\vec{q} \rightarrow \vec{0}$

$$\begin{aligned} Re[K_\sigma] &= 1 + \frac{U}{6N} \sum_{\vec{k}} \Big(\frac{n_F(E^-_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega} + \frac{n_F(E^+_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega} \Big) \\ &= 1 + \frac{U}{3N} \sum_{\vec{k}} n'_F(E^-_{\vec{k}}) + n'_F(E^+_{\vec{k}}) \\ &\neq 0 \end{aligned} \tag{2.41}$$

For vanishing energy and momentum, the excitation still has a finite energy. In other words it has an excitation gap or a mass. We neglect the longitudinal polarized spin waves and focus on the lower energetic transversal fluctuations in the remaining part of the thesis.

2.5.2.2 Kernel of the Berry phase $\vec{\pi}(\vec{n} \times \vec{\pi})$

The treatment of all three kernels is nearly the same. The frequency sum transforms the product of Green's functions into a function of Fermi Dirac distributions

$$\begin{aligned} K_{berry} &= \frac{2U}{3}\frac{1}{N\beta}\sum_{k} g_{1(k)}g_{\tau(k+q)} - g_{\tau(k)}g_{1(k+q)} \\ &= \frac{U}{3}\frac{1}{N\beta}\sum_{k} g_{+(k)}g_{-(k+q)} - g_{-(k)}g_{+(k+q)} \\ &= \frac{U}{3N}\sum_{\vec{k}} \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + i\Omega_n} - \frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + i\Omega_n} \end{aligned} \tag{2.42}$$

The analytical continuation in connection with equation (2.40) separates the real part and the retarded imaginary:

$$Re[K_{berry}] = \frac{U}{3N}P\sum_{\vec{k}} \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega} - \frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega} \tag{2.43}$$

$$\begin{aligned} Im[K_{berry}] &= -\pi\frac{U}{3N}\sum_{\vec{k}} \left[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})\right]\delta_{(E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega)} \\ &\qquad - \left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})\right]\delta_{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega)} \end{aligned} \tag{2.44}$$

The principle value of a sum is not well defined, but later we take the continuum limit, and transform the sum to an integral. Then the principle value is needed. But for now, we just bookkeep it.

We first focus on the dynamics of the system, which is described by the real part of the kernels. This imaginary part, responsible for dissipation, will be discuss in the Chapter 3 in the context of for stability.

Because the energy and momentum of a spin-wave is small compared to an electronic system, we can expand the real part of the kernel to the lowest order dependence of $\vec{q}$ and Ω.

The limit $\vec{q} = 0$ and $\Omega \rightarrow 0$ reveals the lowest order contribution of Ω. (remember $E_k^+ - E_k^- = \Delta$)

$$
\begin{aligned}
Re[K_{berry}] &= \frac{U}{3N} P \sum_{\vec{k}} \big(n_F(E_{\vec{k}}^-) - n_F(E_{\vec{k}}^+)\big)\Big(\frac{-1}{\Delta + \Omega} - \frac{1}{-\Delta + \Omega}\Big) \\
&\approx \frac{2U}{3N} \sum_{\vec{k}} \big(n_F(E_{\vec{k}}^-) - n_F(E_{\vec{k}}^+)\big) \frac{\Omega}{\Delta^2} = \frac{\Omega}{\Delta}
\end{aligned}
\tag{2.45}
$$

The sum is equal to the mean field expression for h_o, which cancels out with Δ, as you can see in equation (2.18) and (2.20).

To derive the lowest order contribution in $\vec{q}$, the limit $\Omega = 0$ and $\vec{q} \rightarrow 0$ is examined. Therefore the summation is shifted to $k' = k - q$ in the second half of equation (2.43).

$$
Re[K_{berry}] \propto P \sum_{\vec{k}} \frac{n_F(E_{\vec{k}}^+) - n_F(E_{\vec{k}+\vec{q}}^-)}{E_{\vec{k}}^+ - E_{\vec{k}+\vec{q}}^-} - P \sum_{\vec{k}'} \frac{n_F(E_{\vec{k}'-\vec{q}}^-) - n_F(E_{\vec{k}'}^+)}{E_{\vec{k}'-\vec{q}}^- - E_{\vec{k}'}^+} \tag{2.46}
$$

If we also let $\vec{q} = 0$, the two sums are the same and cancel each other. Therefore, the Berry phase kernel can never produce a gap, independently of the electron dispersion relation.

So we have to keep a small $\vec{q}$. By changing the sign of the summation index $k' = -k$ in the second part:

$$
Re[K_{berry}] \propto P \sum_{\vec{k}} \frac{n_F(E_{\vec{k}}^+) - n_F(E_{\vec{k}+\vec{q}}^-)}{E_{\vec{k}}^+ - E_{\vec{k}+\vec{q}}^-} - \frac{n_F(E_{-\vec{k}}^+) - n_F(E_{-(\vec{k}+\vec{q})}^-)}{E_{-\vec{k}}^+ - E_{-(\vec{k}+\vec{q})}^-} \tag{2.47}
$$

If the energy is an even function of $\vec{k}$, then $E_{-k} = E_k$, and two sums cancel even with a finite $\vec{q}$. Or in other words the Berry phase kernel vanishes for every order of $\vec{q}$ at $\Omega = 0$.

This property reflects the commutation relation of the opersators for different components of the total spin of all electrons in the system.

As a preview for later in the case of a current, this symmetry is broken, and there will be a $\vec{q}$-dependence.

2.5.2.3 Kernel of the transversal fluctuations $\vec{\pi}$

Again the Matsubara frequency sum is performed for the kernel for the π scalar product.

$$\begin{aligned} K_\pi &= 1 + \frac{2U}{3N\beta}\sum_k g_{1(k)}g_{1(k+q)} - g_{\tau(k)}g_{\tau(k+q)} \\ &= 1 + \frac{U}{3N\beta}\sum_k g_{-(k)}g_{+(k+q)} + g_{+(k)}g_{-(k+q)} \\ &= 1 + \frac{U}{3N}\sum_{\vec{k}}\Big(\frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + i\Omega_n} + \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + i\Omega_n}\Big) \end{aligned} \tag{2.48}$$

The analytical continuation (2.40) divides the dynamical real part, and the dissipativ imaginary part.

$$Re[K_\pi] = 1 + \frac{U}{3N}P\sum_{\vec{k}}\frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega} + \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega} \tag{2.49}$$

$$\begin{aligned} Im[K_\pi] = -\pi\frac{U}{3N}\sum_{\vec{k}} &\Big[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})\Big]\delta_{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega)} \\ &+ \Big[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})\Big]\delta_{(E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega)} \end{aligned} \tag{2.50}$$

As before, the imaginary part is discussed later.

In the $\vec{q} = 0$ and $\Omega \rightarrow 0$ limit of the real part, we obtain the lowest order dependence in Ω.

$$
\begin{aligned}
Re[K_\pi] &= 1 + \frac{U}{3N} P \sum_{\vec{k}} \Big(\frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})}{E^-_{\vec{k}} - E^+_{\vec{k}} + \Omega} + \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}})}{E^+_{\vec{k}} - E^-_{\vec{k}} + \Omega} \Big) \\
&= 1 + \frac{U}{3N} P \sum_{\vec{k}} \big(n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}}) \big) \frac{2\Delta}{\Delta^2 - \Omega^2} \\
&\approx 1 + \frac{2U}{3V} P \sum_{\vec{k}} \big(n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}}) \big) \frac{1}{\Delta} \big(1 + \frac{\Omega^2}{\Delta^2} \big) = -\frac{\Omega^2}{\Delta^2} \qquad (2.51)
\end{aligned}
$$

By replacing the sum with the mean field expression for the auxiliary field, everything cancels out with the $1/\Delta$. So the 1 in front is removed.

Because the Berry phase term already goes linear in Ω, the Ω^2-term is only a correction, and neglected in the further discussion.

We also learn from equation 2.51, that the kernel is vanishing for $\vec{q} = 0$ and $\Omega = 0$. So the transversal excitation is massless.

Now the $\Omega = 0$ and $\vec{q} \rightarrow 0$ limit of equation (2.49) reveals the lowest order dependence in $\vec{q}$. To avoid a $\vec{q}$ dependence of the Fermi Dirac distribution, the summation index is shifted appropriately by $\pm q$.

$$
\begin{aligned}
Re[K_\pi]\Big|_{\Omega=0} &= 1 + \frac{U}{3N} P \sum_{\vec{k}} \; \frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}}} + \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}})}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}}} \\
&= 1 + \frac{U}{3N} P \sum_{\vec{k}} \; n_F(E^-_{\vec{k}}) \Big[\frac{1}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}}} + \frac{-1}{E^+_{\vec{k}-\vec{q}} - E^-_{\vec{k}}} \Big] \\
&\qquad + n_F(E^+_{\vec{k}}) \Big[\frac{-1}{E^-_{\vec{k}-\vec{q}} - E^+_{\vec{k}}} + \frac{1}{E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}}} \Big] \qquad (2.52)
\end{aligned}
$$

To proceed, we need the dispersion relation for the electrons. As the simplest approximation to the problem, we chose the free quadratic dispersion:

$$E^{\pm}_{\vec{k}} = \frac{\hbar^2 k^2}{2m} - \mu \pm \Delta/2 \tag{2.53}$$

Of course this step restrict the validity of the model !

The denominators of the kernel can be replaced by these expressions:

$$E^{-}_{\vec{k}} - E^{+}_{\vec{k}+\vec{q}} = -\frac{\hbar^2}{m}kq\cos\theta - \frac{\hbar^2 q^2}{2m} - \Delta \tag{2.54}$$

$$E^{+}_{\vec{k}-\vec{q}} - E^{+}_{\vec{k}} = -\frac{\hbar^2}{m}kq\cos\theta + \frac{\hbar^2 q^2}{2m} + \Delta \tag{2.55}$$

Because the problem is isotropic in space, we can Taylor expand equation (2.52) in $|q|$, and ignore the terms linear in $\cos\theta$, because they vanish through the angluar integrations.

$$\begin{aligned} Re[K_\pi]\Big|_{\Omega=0} &= 1 + \frac{2U}{3N} P \sum_{\vec{k}} \left[n_F(E^{+}_{\vec{k}}) - n_F(E^{-}_{\vec{k}}) \right] \frac{1}{\Delta} \\ &\quad + \frac{2U}{3N} P \sum_{\vec{k}} \left[n_F(E^{+}_{\vec{k}}) - n_F(E^{-}_{\vec{k}}) \right] \frac{\hbar^4 k^2 \cos^2[\theta]}{m^2 \Delta^3} \cdot q^2 \\ &\quad + \frac{2U}{3N} P \sum_{\vec{k}} \left[n_F(E^{+}_{\vec{k}}) + n_F(E^{-}_{\vec{k}}) \right] \frac{\hbar^2}{2m\Delta^2} \cdot q^2 \\ &\equiv +\frac{\rho}{\Delta} q^2 \end{aligned} \tag{2.56}$$

The constant terms cancels with the same argument as in equation (2.51), so the end result is a gapless excitation.

The spin wave stiffness ρ as defined here, will reappear several times later, so is worth to remember.

2.6 Results

We have derived the action in the lowest orders of Ω and $\vec{q}$:

$$S_{\vec{\pi}}^{(2)} \;=\; \frac{1}{2N\beta\Delta}\sum_q \rho\vec{q}^{\,2}\,\vec{\pi}_{(-q)}\vec{\pi}_{(q)} \;+\; i\Omega\,\vec{\pi}_{(-q)}\big(\vec{n}\times\vec{\pi}_{(q)}\big) \tag{2.57}$$

2.6.1 Magnon Dispersion Relation

To derive the dispersion relation of the ferromagnetic magnons, we write equation (7.14) in a matrix form and choose $\vec{n} = (0,0,1)^T$:

$$S_{\vec{\pi}}^{(2)} \;=\; \frac{1}{2N\beta}\sum_q \frac{1}{\Delta}\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix}\begin{bmatrix}\rho\vec{q}^{\,2} & -i\Omega\\ i\Omega & \rho\vec{q}^{\,2}\end{bmatrix}\begin{bmatrix}\pi_1\\ \pi_2\end{bmatrix} \tag{2.58}$$

The matrix connecting the fluctuating fields is the inverse of the magnetic response function. If we can excite quasi particles the response to an external perturbation diverges, so the eigenvalues λ of the inverse response matrix must become zero.

$$0 \;=\; \lambda_\pm \;=\; \rho\vec{q}^{\,2} \pm \Omega \tag{2.59}$$

It is sufficient to allow only positive frequencies, so we choose the λ_- eigenvalue. The result is the quadratic dispersion relation of ferromagnetic magnons.

2.6.2 Landau Lifschitz Equation

If we Fourier transform the action back in $(t,\vec{r})$ space, we can replace the $\vec{q}$ and Ω by derivatives.

$$S_{\vec{\pi}}^{(2)} \;=\; \frac{1}{2\Delta}\int dt \sum_{\vec{r}} \rho\,\nabla\vec{\pi}\,\nabla\vec{\pi} - \,\vec{\pi}\big(\vec{n}\times\dot{\vec{\pi}}\big) \tag{2.60}$$

By setting the functional derivation of the action equal to zero the classical behavior of the system is recovered,

$$-2\rho\Delta\vec{\pi} - \dot{\vec{\pi}} \times \vec{n} + \vec{n} \times \dot{\vec{\pi}} = 0$$

$$\Rightarrow \dot{\vec{\pi}} = +\rho\Delta\vec{\pi} \times \vec{n} \tag{2.61}$$

which is just the Landau-Lifschitz equation.

2.7 Summary

We have derived the magnon dispersion relation and the Landau Lifschitz equation from the microscopic picture of the Hubbard Model.

The starting point was the Hubbard model. To integrate out the fermions exactly, an auxiliary filed was introduced, which was interpreted as a magnetic field.

The mean field solution of the auxiliary field was derived through functional derivation of the action and the obtained result is similar to Pauli ferromagnetism.

By introducing quadratic fluctuations around this classical ground state we got two possible excitations, a longitudinal excitation with gap, and a gapless excitation called magnons.

We derived an action of the magnons to the lowest order in Ω and $\vec{q}$. By setting the eigenvalues of the kernel equal zero, the quadratic dispersion relation of ferromagnetic magnons was derived, and the classical behavior was recovered by the functional derivation of the action.

Chapter 3

Technical Introduction II: Dissipation, external Fields and Stability

This is another demonstration of the modus operandi. A ferromagnet is stable and the behavior of a magnet in an external field is also known, so we do not expect new physical phenomenons. Nevertheless is is a good exercise to become familiar with the stability conditions, because we will need them in the case of a spin current acting on the system.

A system can have two ways of being instable. The environmental circumstances can make a change of the system necessary or the system is so strongly driven away from the ground state that a new state is favorable, as in a laser.

3.1 Applying an External Field

To examine the first class of instability, an external field $\vec{B} = B\vec{e}_z$ is applied. With this field the action (2.12) changes:

$$\begin{aligned} S &= \frac{1}{N\beta}\sum_q \frac{\vec{h}_{(q)}\vec{h}_{(-q)}}{2} \\ &+ \frac{1}{N\beta}\sum_{kq\sigma\sigma'} \bar{\psi}_{(k,\sigma)}\left[-i\omega_n+\epsilon_k-\mu-\sqrt{\frac{U}{3}}\vec{h}_q\vec{\tau}_{\sigma\sigma'} - g\mu_B\vec{B}\vec{\tau}_{\sigma\sigma'}\right]\psi_{(k+q,\sigma')} \end{aligned} \tag{3.1}$$

And so does the Green's function change.

$$G = \frac{-1}{i\omega_n - \left[\epsilon_k - \mu - (\sqrt{\frac{U}{3}}\vec{h}_q + g\mu_B\vec{B})\vec{\tau}\right]} \tag{3.2}$$

The derivation of the mean field expression for the magnetization has formal the same structure (2.20). But now, the unity vector $\vec{n}$ consists of a sum of magnetization and external field.

$$\begin{aligned}
\vec{h}_o &= \sqrt{\frac{U}{3}}\frac{1}{N}\sum_{\vec{k}}\left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})\right]\vec{n} \\
&= \sqrt{\frac{U}{3}}\frac{1}{N}\sum_{\vec{k}}\left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})\right]\frac{\sqrt{\frac{U}{3}}\vec{h}_o + g\mu_B\vec{B}}{\left|\sqrt{\frac{U}{3}}\vec{h}_o + g\mu_B\vec{B}\right|} \qquad (3.3)
\end{aligned}$$

By considering only the directional degree of freedom of $\vec{h}$, $\vec{B}$ and $\vec{n}$, two possible alignments solve the equation: $\vec{h}$ and $\vec{B}$ can be parallel or anti-parallel. Because of the (anti)parallel alignment, the resulting Zeeman splitting $\widetilde{\Delta}$ between spin up and down electrons is just the (difference)sum of the length of the vectors. The equation for the Zeeman splitting (2.17) becomes

$$\widetilde{\Delta} = \sqrt{\frac{U}{3}}h_o \pm g\mu_B B = \Delta \pm g\mu_B B \tag{3.4}$$

From here, all the calculations can be copied from Chapter 2, until the calculation of the kernels. Because we are expecting a gap for the spin waves [20], we examine the changes of the diagonal kernel K_π (2.51). As mentioned in Chapter 2, the Berry phase kernel can not produce a gap.

$$\begin{aligned}
Re[K_\pi]\Big|_{\vec{q}=0} &= 1 + \frac{U}{3N}P\sum_{\vec{k}}\Big(\frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}})}{E^+_{\vec{k}} - E^-_{\vec{k}} + \Omega} + \frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})}{E^-_{\vec{k}} - E^+_{\vec{k}} + \Omega}\Big) \\
&\approx 1 + \frac{2U}{3V}P\sum_{\vec{k}}\big(n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}})\big)\frac{1}{\widetilde{\Delta}}\big(1 + \frac{\Omega^2}{\widetilde{\Delta}^2}\big) \qquad (3.5)
\end{aligned}$$

In the previous case, the $\vec{q}$ and Ω independent part cancel exactly with the 1 which generated the expected massless excitations. Now the kernel K_π contains $\widetilde{\Delta}$, but the mean field expression of the magnetization (2.20) still depends on Δ. So a constant surplus remains.

$$Re[K_\pi]\Big|_{q=0} \approx 1 - \frac{\Delta}{\widetilde{\Delta}} = 1 - \frac{\Delta}{\Delta \pm g\mu_B B} \tag{3.6}$$

Because the magnetic field inside a ferromagnetic material is quite large compared to easy realizable external fields, we can take B as a weak field compared to the magnetization. Therefore we can Taylor expand the expression:

$$Re[K_\pi]\Big|_{q=0} = 1 - \frac{\Delta}{\Delta \pm g\mu_B B} \approx \pm\frac{g\mu_B B}{\Delta} \tag{3.7}$$

For small fields the a spin wave gap is proportional to the disturbing magnetic field.

In the other limit $K_\pi\big|_{\Omega=0}$ the changed Zeeman splitting will change the spin stiffness ρ, but the change is not so dramatic. And the kernel K_{berry} is not affected at all, if we remove the overall factor $1/\widetilde{\Delta}$.

To determine the excitations of the system we write the action in matrix form

$$S^{(2)}_{\vec{\pi}} = \frac{1}{2N\beta}\sum_q \frac{1}{\widetilde{\Delta}} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \begin{bmatrix} \rho\vec{q}^2 \pm g\mu_B B & -i\Omega \\ i\Omega & \rho\vec{q}^2 \pm g\mu_B B \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \tag{3.8}$$

and set the physical eigenvalue λ (for positive frequencies) of the matrix equal zero.

$$\lambda_+ = 0 \rightarrow \Omega = \rho\vec{q}^2 \pm g\mu_B B \tag{3.9}$$

Now something very interesting happens. If the magnetization is anti-parallel to the external field, then the dispersion relation is shifted down.

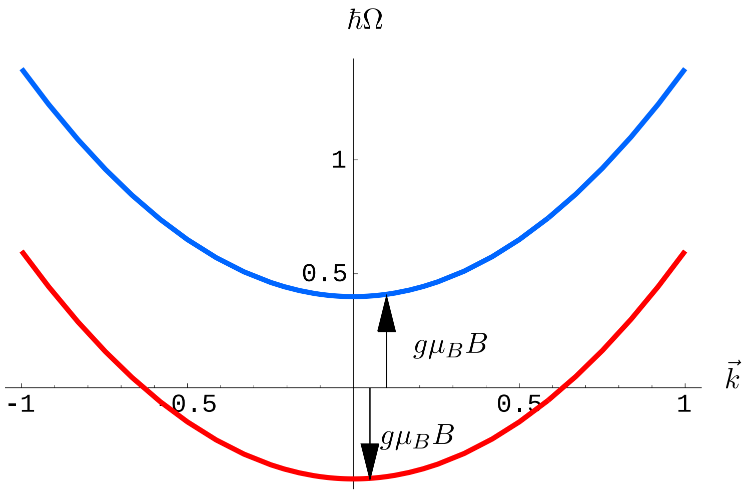


Figure 3.1: Magnon dissipation relation with external field

This means that at $\Omega = 0$ we can create magnons with a certain momentum, but without paying energy. Such a dispersion relation describes a system which is not stable. We know, that a ferromagnet want to be aligned parallel to the external field, so this is a familiar case of an instability.

If we Fourier transform the expression, and derive the Landau-Lifschitz equation, we obtain the calssical expression:

$$\dot{\vec{\pi}} = \left(+\rho\Delta\vec{\pi} - g\mu_B\vec{B}\right) \times \vec{n} \tag{3.10}$$

3.2 Dissipation and Stability

There is another way to make the system unstable. If you perturb the system it may become energetically favorable for the system to create excitations, as in a laser. There if level inversion is strong enough, the system starts to lase. Instead of damping the excitations, which would mean exponentially decreasing the photon field energy, the field energy rises exponentially. So the sign of the dissipation term decides about stability. The criterion for the stability is derived in the book by Forster [21]

$$\sum_{ij} a_i^{\star} \Omega \chi''_{ij} a_j \geq 0 \tag{3.11}$$

Where χ''_{ij} is the imaginary part of the susceptibility, and i,j are the super indices. Because the a's are arbitrary this is equal to the sum of all the imaginary matrix entries. Forster derives this condition out of the conservation of total energy of the system. If the criterion (3.11) is not satisfied, we do not have the true ground state.

The magnetic susceptibility can be obtained from the inveres spin wave Green's function (2.58).

$$K_{ij}^{-1} = \chi_{ij} \tag{3.12}$$

This is a matrix equation, which can be delt with using the same procedure as in equation (2.17).

$$\begin{aligned} K_{ij}^{-1} = \left[K_\pi + K_{berry} \tau^y \right]^{-1} &= \frac{1}{2}\Big(\frac{1}{K_\pi + K_{berry}} + \frac{1}{K_\pi - K_{berry}}\Big) \\ &+ \frac{1}{2}\Big(\frac{1}{K_\pi + K_{berry}} - \frac{1}{K_\pi - K_{berry}}\Big)\tau^y \end{aligned} \tag{3.13}$$

Due to the different signs of the off diagonal elements, the term proportional to the Pauli matrix vanishes in the sum. Only the imaginary part of the diagonal entries are relevant for stability:

$$\begin{aligned} \chi''_{ii} &= -\frac{1}{2\big|K_\pi + K_{berry}\big|^2} Im[K_\pi + K_{berry}] \\ &\quad -\frac{1}{2\big|K_\pi - K_{berry}\big|^2} Im[K_\pi - K_{berry}] \end{aligned} \tag{3.14}$$

We are looking for exponential behavior, therefore we search for a contribution linear in ω. To find this we evaluate the $\vec{q} = 0$ limit by using equation (2.44) and (2.50) for the kernels.

$$\begin{aligned} \chi''_{ii}\Big|_{\vec{q}=0} &= +\frac{\pi U}{3N\left|K_\pi + K_{berry}\right|^2} \sum_{\vec{k}} \left[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}})\right] \delta_{(\Omega+\Delta)} \\ &\quad +\frac{\pi U}{3N\left|K_\pi - K_{berry}\right|^2} \sum_{\vec{k}} \left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})\right] \delta_{(\Omega-\Delta)} \end{aligned} \tag{3.15}$$

If we restrict Ω to be positive, only the second term of equation (3.15) contributes. Because the Fermi distribution is a monotonically decreasing function χ''_{ii} is positive.

As expected, the system is stable. The dissipation term has a simple physical interpretation. Only if the magnon energy is exactly the Zeeman energy splitting the bands, then the magnons can be absorbed by an electron. This electron has to flips its spin and jumps from the spin down occupied to an upper not occupied spin up state.

Chapter 4

Including a Current

Now begins the interesting part of this thesis. We will include a current in the model, but without enforcing a special spin structure of the current.

4.1 Hamiltonian with constraint

The starting point is the already Hubbard-Stratonowich transformed Hamiltonian

$$H = \sum_{\vec{k}\sigma} \epsilon_{(\vec{k})}\ c^{\dagger}_{\vec{k}\sigma} c_{\vec{k}\sigma} - \sqrt{\frac{U}{3}} \sum_{\vec{k}\vec{q}\sigma\sigma'} \vec{h}_{(\vec{q})}\, \vec{\tau}_{\sigma\sigma'}\ \ c^{\dagger}_{\vec{k}\sigma} c_{\vec{k}+\vec{q}\sigma'} + \sum_{\vec{q}} \frac{\vec{h}_{(\vec{q})}\vec{h}_{(-\vec{q})}}{2} \tag{4.1}$$

To include a current in the problem, the partition function get a constraint.

$$\vec{C}_{(q)} = \sum_{\vec{k}\sigma} \vec{v}_{(q)}\ c^{\dagger}_{k\sigma} c_{k+q\sigma}\ - \vec{I}_{(q)}\ = 0 \tag{4.2}$$

With $\vec{v}_{(q)} = \frac{\hbar}{2m}(2\vec{k}+\vec{q})$. This constraint removes every state out of the partition function which does not provide the required current $\vec{I}_{(q)}$.

Because of the possible time dependence of the condition, it must be enforced at every (imaginary) time slice n. Therefore we write the condition as a product of delta functions.

$$Z < \delta(\vec{C}) > \ = \ \int D\bar{\psi}\, D\psi\, D\vec{h}\ e^{-S(\psi,\bar{\psi},\vec{h})} \prod_{n=1}^{N} \delta(\vec{C}_{(n\cdot d\tau)}) \tag{4.3}$$

The delta function can be represented as an integral over a dummy variable $\vec{B}$, and the product of integrals can be rewritten as a path integral.

$$\begin{aligned} Z < \delta(\vec{C}) > &= \int D\bar{\psi}\, D\psi\, D\vec{h}\; e^{-S(\psi,\bar{\psi},\vec{h})} \prod_{n=1}^{N} \int d\vec{B}_{(n\cdot d\tau)} e^{-i\vec{B}_{(n\cdot dt)}\vec{C}_{(n\cdot d\tau)}} \\ &= \int D\bar{\psi}\, D\psi\, D\vec{h}\; e^{-S(\psi,\bar{\psi},\vec{h})} \int D\vec{A}\; e^{-\int d\tau \vec{A}_{(\tau)}\vec{C}_{(\tau)}} \end{aligned} \tag{4.4}$$

We introduce the new variable $\vec{A} = i\vec{B}$, because in the following physical context, $\vec{B}$ would become purely imaginary. But to interpret these variable as a physical field, and to reduce bookkeeping we prefer a real veriable $\vec{A}$. Now the constraint term has the same shape as the action and can be absorbed in the effective action.

$$Z = \int D\bar{\psi}\, D\psi\, D\vec{h}\, D\vec{A}\; exp[-S'] \tag{4.5}$$

with the altered effective action.

$$\begin{aligned} S' &= \frac{1}{N\beta}\sum_{q} \vec{A}_{(q)}\vec{I}_{(q)} + \frac{h_{(-q)}h_{(q)}}{2} \\ &+ \frac{1}{N\beta}\sum_{k\,q,\sigma\sigma'} \bar{\psi}_{(k,\sigma)}(-i\omega_n - \mu + \epsilon_{\vec{k}} - \vec{A}\vec{v}_{\vec{q}} - \sqrt{\frac{U}{3}}\vec{\tau}\vec{h}_q)\psi_{(k+q,\sigma')} \end{aligned} \tag{4.6}$$

The structure of $\vec{A}$ is in two aspects very interesting. From the mathematical point of view it looks like a Lagrange multiplier. And for a physicist it has the same purpose as a electromagnetic vector potential. But because we are imposing a particle current, and not an electrical, $\vec{A}$ has the units of a momentum.

The constraint is also only quadratic in the Grassman fields, so the fermions can be integrated out exactly.

$$S_{eff} = \frac{1}{N\beta}\sum_q \vec{A}_{(q)}\vec{I}_{(q)} + \frac{\vec{h}_{(-q)}\vec{h}_{(q)}}{2} - Tr\ln[G^{-1}] \tag{4.7}$$

with the Green's function.

$$G^{-1} = -i\omega_n - \mu + \epsilon_{\vec{k}} - \vec{A}\vec{v}_{\vec{k}\vec{k}'} - \sqrt{\frac{U}{3}}\vec{\tau}\vec{h}_{\vec{k}\vec{k}'} \tag{4.8}$$

The main effect of the constraint is to insert the $\vec{A}\vec{v}_{\vec{q}}$ term in the inverse of the Green's function. To shorten the expressions we define the same notation as in equation (2.18), and using the tilde to indicate the existence of the $\vec{A}\vec{v}_{\vec{q}}$ term.

$$\begin{aligned} \widetilde{E}^{\pm}_{k} &= \epsilon_k - \mu - \vec{A}\vec{v}_{\vec{k}} \pm \Delta/2 \\ g_{\pm} &= \frac{1}{i\omega_n - \widetilde{E}^{\pm}_{k}} \end{aligned} \tag{4.9}$$

4.2 Extremizing the action

The classical solution for the Lagrange multiplier extremizing the action, therefore it can be determined by setting the functional derivative of the action equal zero.

$$\begin{aligned} 0 &= \frac{\delta S_{eff}}{\delta \vec{A}_{(q_o)}} = \vec{I}_{(q_o)} - \frac{\partial}{\partial \vec{A}_{(q_o)}} Tr\ln[G^{-1}] \\ &\rightarrow \quad \vec{I}_{(q_o)} = -\frac{1}{N\beta}\sum_{k\sigma\sigma'}\langle k,\sigma|\ G\ |k+q_o,\sigma'\rangle \vec{v}_{(-q_o)} \end{aligned} \tag{4.10}$$

And the classical magnetization can be derived at the very same way as in equation (2.16).

$$h^a_{(q_o)} = +\sqrt{\frac{U}{3}}\frac{1}{N\beta}\sum_{k\sigma}\langle k,\sigma|\, G\,\tau^a\,|k+q_o,\sigma\rangle \tag{4.11}$$

These two equations form a non trivial system. To describe a physical system with a certain current $I_{(q)}$ we want to solve the equation (4.10) for the necessary vector potential. But the Green's function contains $\vec{h}_{(q)}$. And on the other side to calculate the magnetization we need the vector potential, which is also included in the Green's function.

To continue the analytical calculation, we choose the magnetization $\vec{h}$ and the current $\vec{I}$ as constant in space and time. The choice of the current is justified by the experiments done on this systems, and the choice of the static field is proven by the stability of the system.

With this choice all 4-momentum traces becomes diagonal, and we can perform the Matsubara frequency sums.

$$\vec{h} = -\sqrt{\frac{U}{3}}\frac{1}{N}\sum_{\vec{k}}\left[n_F(\widetilde{E}^+_{\vec{k}}) - n_F(\widetilde{E}^-_{\vec{k}})\right]\vec{n} \tag{4.12}$$

In the trace for the current is no additional Pauli matrix. So the spin trace will select the diagonal part of the Green's function instead of the spin dependent in the case of the magnetic field. This causes plus sign between the spin up and down Green's function and after the frequency sum between the distributions.

$$\vec{I} = +\frac{1}{N}\sum_{\vec{k}}\left[n_F(\widetilde{E}^+_{\vec{k}}) + n_F(\widetilde{E}^-_{\vec{k}})\right]\vec{v} \tag{4.13}$$

The minus sign is canceled with the minus included in G. If we assume parabolic bands, then the square in the argument of the Fermi distribution can be completed. To simplify notation, without loosing the correct units, I define $\vec{A} = \hbar\vec{a}$, with the wave vector $\vec{a}$.

$$\begin{aligned} \widetilde{E}^{\pm}_{\vec{k}} &= \frac{\hbar^2\vec{k}^2}{2m} - \vec{A}\frac{\hbar^2\vec{k}}{m} - \mu \pm \Delta/2 \\ &= \frac{\hbar^2}{2m}(\vec{k}-\vec{a})^2 - \frac{\hbar^2\vec{a}^2}{2m} - \mu \pm \Delta/2 = E^{\pm}_{\vec{k}-\vec{a}} - \epsilon_{\vec{a}} \end{aligned} \tag{4.14}$$

By shifting the momentum index in equation 2.20 to $\vec{k}' = \vec{k} - \vec{a}$, the summations becomes symmetric over a sphere.

$$\vec{I} = \frac{1}{N}\sum_{\vec{k}'}\left[n_F(E^{+}_{\vec{k}'} - \epsilon_{\vec{a}}) + n_F(E^{-}_{\vec{k}'} - \epsilon_{\vec{a}})\right]\left(\frac{\hbar\vec{k}'}{m} + \frac{\hbar\vec{a}}{m}\right) \tag{4.15}$$

Because of the symmetry of the integration interval, the term linear in $\vec{k}$ vanishes, and the remaining sum only counts N_e = the number of the electrons.

$$\vec{I} = +\frac{N_e}{N}\frac{\vec{A}}{m} \tag{4.16}$$

With the $1/N$ vanished, if we transform it in the real space, and divided by the Volume, it looks just like the current density derived by a Drude model.

$$\vec{j} = n\frac{\vec{A}}{m} = n\vec{v} \tag{4.17}$$

4.3 Missing Gauge Invariance

As you can see, in equation 4.14, we can not transform $\widetilde{E}^{\pm}_{\vec{k}} \neq E^{\pm}_{\vec{k}+\vec{a}}$, this means our theory is not gauge invariant.

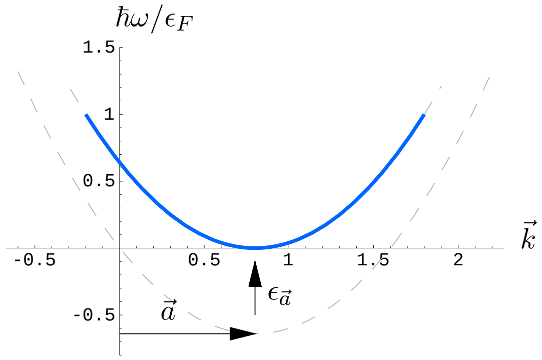


Figure 4.1: Renormalization of the chemical potential

This flaw arises from the technique, to introduce the vector potential in equation (4.4). To solve this problem is an exact mathematical way, we have to include an additional auxillary field F, to ensure the conservation of the particle number N_e under the influence of a current. So the partition function looks like:

$$Z = \int D\bar{\psi}\, D\psi\, D\vec{h}\, D\vec{A}\, DF\; exp[-S'] \tag{4.18}$$

with the altered effective action.

$$\begin{aligned} S' &= \frac{1}{N\beta}\sum_q \vec{A}_{(q)}\vec{I}_{(q)} - N_e\, F_{(q)} + \frac{h_{(-q)}h_{(q)}}{2} \\ &+ \frac{1}{N\beta}\sum_{k\,q,\sigma\sigma'} \bar{\psi}_{(k,\sigma)}(-i\omega_n - \mu + \epsilon_{\vec{k}} - \vec{A}\vec{v}_{\vec{q}} + F_{(q)} - \sqrt{\frac{U}{3}}\vec{\tau}\vec{h}_q)\psi_{(k+q,\sigma')} \end{aligned} \tag{4.19}$$

And as we know, the particke number is conserved for $F = \vec{A}^2/2$. Therefore the additional term $N_eF_{(q)} = N_e \cdot \vec{A}^2$ becomes the diamagnetic current after the functional derivative with respect to $\vec{A}_{(q_o)}$.

Because this is not an essential part of the theory, we simply ignore the additional energy shift $\epsilon_{\vec{a}}$, and renormalize the chemical potential. Due to this crude procedure, we can avoids a reasonable amount of bookkeeping.

This crude method is justified, because outside the Fermi distributions, the energy appears only in differences, and there the difference vanishes.

$$\widetilde{E}^{\pm}_{\vec{k}} - \widetilde{E}^{\mp}_{\vec{k}} = E^{\pm}_{\vec{k}+\vec{a}} - E^{\mp}_{\vec{k}+\vec{a}} \tag{4.20}$$

4.4 Quadratic fluctuations with current

As long as the constraint is absorbed in g_1 and g_τ, and in g_+ and g_-, we can copy the calculation from the case without constraints. The first change occurs in the calculation of the kernels. All kernels consist of terms like

$$\sum_{\vec{k}} \frac{n_F(\widetilde{E}^{+}_{\vec{k}}) - n_F(\widetilde{E}^{-}_{\vec{k}+\vec{q}})}{\widetilde{E}^{+}_{\vec{k}} - \widetilde{E}^{-}_{\vec{k}+\vec{q}} + \Omega} \tag{4.21}$$

By completing the square

$$\sum_{\vec{k}} \frac{n_F(E^{+}_{\vec{k}+\vec{a}} - \epsilon_{\vec{a}}) - n_F(E^{-}_{\vec{k}+\vec{a}+\vec{q}} - \epsilon_{\vec{a}})}{E^{+}_{\vec{k}+\vec{a}} - E^{-}_{\vec{k}+\vec{a}+\vec{q}} + \Omega} \tag{4.22}$$

and shifting the summation index from $\vec{k}$ to $\vec{k}' = \vec{k} - \vec{a}$,

$$\sum_{\vec{k}'} \frac{n_F(E^{+}_{\vec{k}'} - \epsilon_{\vec{a}}) - n_F(E^{-}_{\vec{k}'+\vec{q}} - \epsilon_{\vec{a}})}{E^{+}_{\vec{k}'} - E^{-}_{\vec{k}'+\vec{q}} + \Omega} \tag{4.23}$$

we see, the only change of the physical system is a shift of the chemical potential. And as discussed in the previous section this is a flaw of the description, and not physically motivated. Therefore we ignore the shift.

Now every influence of the current is lost, and we have reproduced the current free case.

4.5 Result

As a result we can say, that if we let the system choose the spin polarization, the collective excitations are not changed, and we do not see any of the interesting experimental effects.

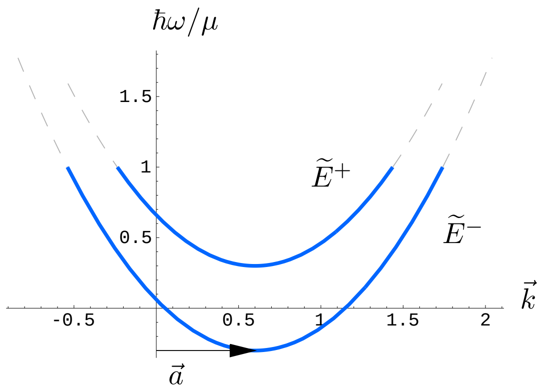


Figure 4.2: Electron dispersion relation with current

On the other hand, the dispersion relation shows, that the current going through the the magnet is already polarized. Without enforcing a polarization, the current chooses a certain polarization to adapt to the enviroment.

Chapter 5

Including a Spin-current

As we have seen, if we apply a vector potential acting on spin up and down electrons the same way, we get a current with a spin structure. Nevertheless, this composition seems to fit to the magnet, so the magnetization is not influenced by this kind of current.

The next approach is to impose a current, with a different spin polarization composition.

5.1 Hamiltonian with constraint

Again the starting point is the Hubbard Stratonovich transformed Hamiltonian (4.1), and now we impose a spin dependent constraint.

$$\vec{C}_{(q)} = \sum_{\vec{k}\sigma\sigma'} \vec{v}_q \left(\vec{n}_c \cdot \vec{\tau}_{\sigma\sigma'}\right) c^{\dagger}_{k\sigma} c_{k+q\sigma'} \; - \vec{I}_{(q)} \; = 0 \tag{5.1}$$

This describes a spin current, polarized in the direction $\hat{n}_c$. We again include the constraint in the path integral and absorb it in the effective action, just like in equation (4.5).

$$Z \;\; = \;\; \int D\bar{\psi}\, D\psi\, D\vec{h}\, D\vec{A}\; exp[-S'] \tag{5.2}$$

with the altered effective action.

$$
\begin{aligned}
S' &= \frac{1}{N\beta}\sum_{q}\vec{A}_{(q)}\vec{I}_{(q)} + \frac{h_{(-q)}h_{(q)}}{2} \\
&+ \frac{1}{N\beta}\sum_{k\,q,\sigma\sigma'}\bar{\psi}_{(k,\sigma)}(-i\omega_n - \mu + \epsilon_{\vec{k}} - (\vec{A}\vec{v}_{\vec{q}})\vec{n}_c\vec{\tau} - \sqrt{\frac{U}{3}}\vec{\tau}\vec{h}_q)\psi_{(k+q,\sigma')}
\end{aligned} \quad (5.3)
$$

Now $\vec{A}$ has a odd physical interpretation. It is a electromagnetic vector potential, which acts reversed on different spin orientations. In this case it makes more sense to focus on the function as a Lagrange multiplier. We impose a purely spin polarized current, and the system reacts on it like acting on this vector potential. We integrate out the fermions:

$$
S_{\mathit{eff}} = \frac{1}{N\beta}\sum_{q}\vec{A}_{(q)}\vec{I}_{(q)} + \frac{\vec{h}_{(-q)}\vec{h}_{(q)}}{2} - Tr\ln[G^{-1}] \quad (5.4)
$$

and obtain this Green's function.

$$
G^{-1} = -i\omega_n - \mu + \epsilon_{\vec{k}} - \big(\sqrt{\frac{U}{3}}\vec{h}_{\vec{k}\vec{k}'} + (\vec{A}\vec{v}_{\vec{k}\vec{k}'})\vec{n}_c\big)\vec{\tau} \quad (5.5)
$$

It is just like the case of the external magnetic field (3.2). Only the external field is now has now a k-dependent effective magnetic field.

5.2 Extremizing the action

To derive the classical fields, we set the functional derivative of the action equal to zero. This leads to the following system of equations:

$$
\begin{aligned}
\vec{I}_{(q_o)} &= -\frac{1}{N\beta}\sum_{k\sigma\sigma'}\langle k,\sigma|\, G\,\vec{\tau}\vec{n}_c|k+q_o,\sigma'\rangle\vec{v}_{(-q_o)} \\
\vec{h}_{(q_o)} &= -\sqrt{\frac{U}{3}}\frac{1}{N\beta}\sum_{k\sigma}\langle k,\sigma|\,G\,\vec{\tau}\,|k+q_o,\sigma\rangle
\end{aligned} \quad (5.6)
$$

To make the system analytically solvable, I choose $\vec{h}$ and $\vec{I}$ as constant in space and time.

With only the $q = 0$ component contributing, all 4-momentum traces become diagonal. For the magnetization we obtain the same as in the case of the external magnetic field (3.3).

$$\vec{h} \quad = \quad +\sqrt{\frac{U}{3}}\frac{1}{N}\sum_{\vec{k}}\left[n_F(\widetilde{E}^-_{\vec{k}}) - n_F(\widetilde{E}^+_{\vec{k}})\right]\frac{\sqrt{\frac{U}{3}}\vec{h} + (\vec{A}\vec{v})\vec{n}_c}{|\sqrt{\frac{U}{3}}\vec{h} + (\vec{A}\vec{v})\vec{n}_c\,|} \tag{5.7}$$

As in the case of the external field, there are only two possible solutions. The magnetization must be parallel or anti-parallel to the polarization direction of the applied current. Let us assume, they are parallel, and the energies add. We can introduce nearly the same short notation for the particle energy:

$$\widetilde{E}^{\pm}_k \quad = \quad \epsilon_k - \mu \pm (\vec{A}\vec{v} + \Delta/2) \tag{5.8}$$

The dispersion relation is plotted in figure 5.1:

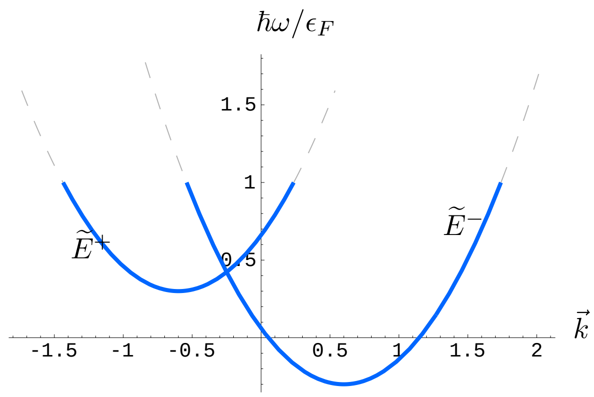


Figure 5.1: Electron dispersion relation with spin current

In contrast to the unpolarized current, there is now an additional Pauli matrix inside the trace. Therefore the spin trace will select the spin dependent part of the Green's function, and this generates a minus sign between the spin up and down Green's function.

$$\vec{I} \;=\; +\frac{1}{N}\sum_{\vec{k}}\left[n_F(\widetilde{E}^{-}_{\vec{k}}) - n_F(\widetilde{E}^{+}_{\vec{k}})\right]\vec{v} \tag{5.9}$$

If we assume parabolic bands, then we can complete the square in the argument of the Fermi distribution. (Remember $\vec{A} = \hbar\vec{a}$)

$$\begin{aligned}\widetilde{E}^{\pm}_{\vec{k}} &= \frac{\hbar^2\vec{k}^2}{2m} \pm \vec{A}\frac{\hbar\vec{k}}{m} - \mu \pm \Delta/2 \\ &= \frac{\hbar^2}{2m}(\vec{k}\pm\vec{a})^2 - \frac{\hbar^2\vec{a}^2}{2m} - \mu \pm \Delta/2 = E^{\pm}_{\vec{k}\pm\vec{a}} - \epsilon_{\vec{a}}\end{aligned} \tag{5.10}$$

And by shifting the momentum summation in to $\vec{k}' = \vec{k} - \vec{a}$, we can sum over a sphere symmetric to the origin, where we again ignore the $\epsilon_{\vec{a}}$ shifts in the chemical potential.

$$\vec{I} \;=\; \frac{1}{N}\sum_{\vec{k}'} n_F(E^{-}_{\vec{k}'})\left(\frac{\hbar\vec{k}'}{m} + \frac{\hbar\vec{a}}{m}\right) - n_F(E^{+}_{\vec{k}'})\left(\frac{\hbar\vec{k}'}{m} - \frac{\hbar\vec{a}}{m}\right) \tag{5.11}$$

The term linear in $\vec{k}$ vanishes, and the remaining sum only counts the number of the electrons.

$$\vec{I} \;=\; \frac{N_e}{N}\frac{\vec{A}}{m} \tag{5.12}$$

This is exactly the same result as in the unpolarized current case. Notice, this is not the overall spin current, it is the variation of the unpolarized current, as defined in the beginning of the chapter.

5.3 How to simulate a real current

As we have seen from the dispersion relations, the vector potential acting equal on the spins produces a current and also produce a spin current, and the spin dependent potential alike.

To simulate a current with an amperage independent from the spin current, we have to apply both Lagrange multipliers $\vec{A}_s$ and $\vec{A}_c$ at the same time. Then the Green's function looks like:

$$G^{-1} = -i\omega_n - \mu + \epsilon_{\vec{k}} - \vec{A}_c\vec{v} - \big(\sqrt{\frac{U}{3}}\vec{h} + (\vec{A}_s\vec{v})\vec{n}_c\big)\vec{\tau} \tag{5.13}$$

We can derive the particle current I_c, by the functional derivative of the action with respect to $\vec{A}_c$ (4.13).

$$\vec{I}_c = \frac{1}{N}\sum_{\vec{k}}\big[n_F(\widetilde{E}^+_{\vec{k}}) + n_F(\widetilde{E}^-_{\vec{k}})\big]\vec{v} \tag{5.14}$$

where $\widetilde{E}^{\pm}_k = \epsilon_k - \mu - \vec{A}_c\vec{v} \pm (\vec{A}_s\vec{v} + \Delta/2)$. Now we can complete the square, and neglect the shift of the chemical potential:

$$\begin{aligned}
\vec{I}_c &= \frac{1}{N}\sum_{\vec{k}}\big[n_F(E^-_{\vec{k}-\vec{a}_c-\vec{a}_s}) + n_F(E^+_{\vec{k}-\vec{a}_c+\vec{a}_s})\frac{\hbar\vec{k}}{m} \\
&= \frac{1}{N}\sum_{\vec{k}'} n_F(E^-_{\vec{k}})(\frac{\hbar\vec{k}}{m} + \frac{\hbar\vec{a_c}}{m} + \frac{\hbar\vec{a_s}}{m}) + n_F(E^+_{\vec{k}})(\frac{\hbar\vec{k}}{m} + \frac{\hbar\vec{a_c}}{m} - \frac{\hbar\vec{a_s}}{m}) \\
&= \frac{N_e}{N\,m}A_c + \frac{N_\downarrow - N_\uparrow}{N\,m}A_s
\end{aligned} \tag{5.15}$$

The analog calculation for the spin current,

$$\begin{aligned}
\vec{I}_s &= \frac{1}{N}\sum_{\vec{k}'}\big[n_F(E^-_{\vec{k}-\vec{a}_c-\vec{a}_s}) - n_F(E^+_{\vec{k}-\vec{a}_c+\vec{a}_s})\frac{\hbar\vec{k}}{m} \\
&= \frac{1}{N}\sum_{\vec{k}'} n_F(E^-_{\vec{k}})(\frac{\hbar\vec{k}}{m} + \frac{\hbar\vec{a_c}}{m} + \frac{\hbar\vec{a_s}}{m}) - n_F(E^+_{\vec{k}})(\frac{\hbar\vec{k}}{m} + \frac{\hbar\vec{a_c}}{m} - \frac{\hbar\vec{a_s}}{m}) \\
&= \frac{N_\downarrow - N_\uparrow}{N\,m}A_c + \frac{N_e}{N\,m}A_s \qquad (5.16)
\end{aligned}$$

with the convention spin down as the majority spins. So to simulate a current with a defined degree of polarization and a defined amperage, we can solve this linear system of equations for the right combination of Lagrange parameters.

5.4 Quadratic fluctuations with current

We again just copy the calculation from Chapter 2 until the calculation of the kernels. And now there are important differences. This is so interesting, that I dedicated the whole next chapter for the discussion of these changes.

Chapter 6

Effects of the Spin Current

We again go through the calculations as in Chapter 2, up to the calculation of the kernels. By replacing $E^{\pm}_{\vec{k}}$ with $\widetilde{E}^{\pm}_{\vec{k}}$, we obtain interesting changes. Because now the denominator of every kernel looks like

$$\frac{1}{\widetilde{E}^{\pm}_{\vec{k}} - \widetilde{E}^{\mp}_{\vec{k}}} = \frac{1}{\pm\Delta \pm \frac{\hbar^2 \vec{A}\vec{k}}{m}} \tag{6.1}$$

And this additional k-dependence of the kernels causes various effects:

6.1 Enhanced Dissipation at critical current

Due to the analytical continuation (2.44,2.50), the modified denominators become the arguments of δ-functions. Therefore the imaginary parts of the kernels change and with them the dissipation of the system. Both kernels consist of these sums:

$$\begin{aligned} \sum_{\vec{k}} & \left[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{q}}) \right] \delta_{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} + \Omega)} \\ & \mp \left[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}+\vec{q}}) \right] \delta_{(E^+_{\vec{k}} - E^-_{\vec{k}+\vec{q}} + \Omega)} \end{aligned} \tag{6.2}$$

Because the momentum of a spin wave is much smaller than of an electron in a degenerate electron gas, we can evaluate the sums in the limit $\vec{q} = 0$.

In the current free case, the argument of the δ-functions could be reduced to $\Delta \pm \Omega$. So only those magnons, which have the same energy as the Zeeman splitting could be absorbed, as you can see in figure 6.1A. This is physical not very interesting, because the typical magnon energy some magnitudes smaller. So the scattering processes need an impurity, to absorb or provide momentum.

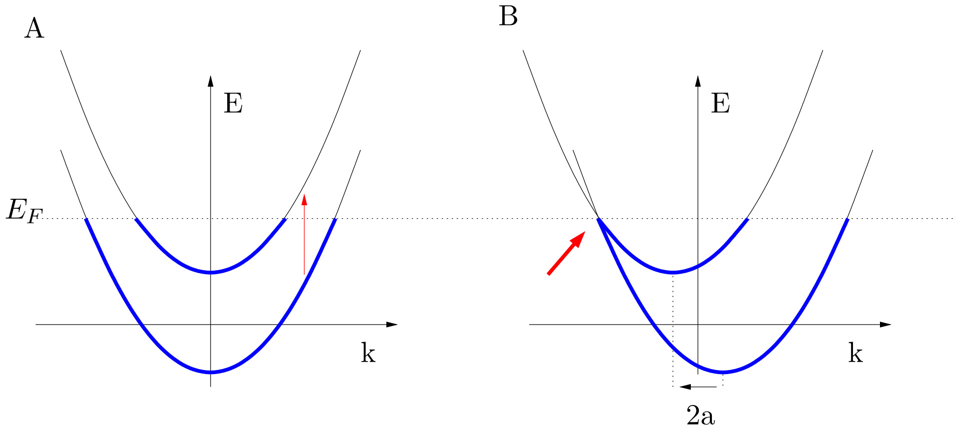


Figure 6.1: Decreasing of the Zeeman splitting

In the case of a spin current, the electron dispersion relations are shifted against each other by $2a$. At a critical current the surfaces of the majority and minority Fermi spheres begin to touch each other, as in figure 6.1B. At this crossing point, the Zeeman splitting of spin up and down band vanishes.

The system can now absorb magnons, which have an energy of nearly zero, on a typical energy scale of electronic excitations. This is a new way to absorb or emit magnons. The condition for the current to enable the crossing is:

$$k_F^+ + a = k_F^- - a \qquad \text{which is equal to} \qquad \widetilde{E}^+_{k_F} = \widetilde{E}^-_{k_F} \tag{6.3}$$

With (5.12) we can calculate the critical current density and compare it with the resistivity steps measured by Tsoi [2] and Myers [1].

$$j = \frac{n}{\sqrt{2m}}\Big(\sqrt{\mu + \frac{\Delta}{2}} - \sqrt{\mu - \frac{\Delta}{2}}\Big) \tag{6.4}$$

If we use typical values for Cobalt ($\mu \approx 7eV$, $n = 9 \cdot 10^{22}\frac{1}{cm^3}$, $\Delta \approx 1eV$ [22]), we obtain a critical current density of $8 \cdot 10^{10}\frac{A}{cm^2}$

The experiments mentioned in the Introduction show an abrupt increase of the resisonly at a current densidy of about $0.75 \cdot 10^9 \frac{A}{cm^2}$ (Tsoi), which is about two magnitudes to small. So it is unlikely, that this is a related effect.

But by repeating the same experiments with magnetic semiconductors, which have a much smaller Zeeman splitting and therefore a much smaller critical current density, this effect may become observable.

The mathematical calculation of the imaginary parts of the kernels (6.2) in the $\vec{q} = 0$ limit for 3 dimensions is explained in Appendix B. For the physical interesting interval $k_F^- - k_F^+ < 2A < k_F^- + k_F^+$ the sum is without further approximations:

$$\sum_{\vec{k}} \Big[n_F(\widetilde{E}^-_{\vec{k}}) - n_F(\widetilde{E}^+_{\vec{k}})\Big]\delta_{(\widetilde{E}^-_{\vec{k}} - \widetilde{E}^+_{\vec{k}} + \hbar\Omega)} = \frac{m^2 V}{8\pi^2\hbar^3|A|}\,\Omega \tag{6.5}$$

which is exactly the same result as for the second sum :

$$\sum_{\vec{k}} \Big[n_F(\widetilde{E}^+_{\vec{k}}) - n_F(\widetilde{E}^-_{\vec{k}})\Big]\delta_{(\widetilde{E}^+_{\vec{k}} - \widetilde{E}^-_{\vec{k}} + \hbar\Omega)} = \frac{m^2 V}{8\pi^2\hbar^3|A|}\,\Omega \tag{6.6}$$

The imaginary part of the Berry phase is proportional to the difference of the sums (2.44), and therefore equal to zero.

$$Im[K_{berry}] \quad = \quad 0 \tag{6.7}$$

And the diagonal kernel K_π (2.50) is the sum of both:

$$Im[K_\pi] = -2\pi \frac{U}{3N} \frac{m^2 V}{8\pi^2 \hbar^3 |A|} \Omega \tag{6.8}$$

This linear dependence is not valid for the small interval of switching on and off. But the width of these intervals scale with Ω, and are vanishing in the limit $\Omega \to 0$. A carefull calculation of all this intervals is in Appendix C.

If you enter this result in the calculation of the susceptibility (3.15), the imaginary part of the susceptibility does not become negative. So we have not a LASER like instability, like other predictions ([8]).

6.2 Modified Magnon Dispersion Relation

We have to calculate the kernels new, now with the modified particle energy (5.8). Because the spin waves have diminutive energy and momentum compared to the typical electron in a degenerate gas, we can Taylor expand in Ω and $\vec{q}$.

$$K_\pi \quad \approx \quad K_\pi\big|_{q=0} + \partial^2_{\vec{q}} K_\pi\big|_{q=0} \cdot \frac{q^2}{2} \tag{6.9}$$

$$K_{Berry} \quad \approx \quad \partial_\Omega K_{Berry}\big|_{q=0} \cdot \Omega + \partial_{\vec{q}} K_{Berry}\big|_{q=0} \cdot q \tag{6.10}$$

The exact result of the calculation is explained in Appendix C. There we introduced $\Delta' = \frac{m\Delta}{2\hbar^2}$ and $\Omega' = \frac{m\Omega}{2\hbar}$ to remove the multiple fraction expressions.

6.2.1 $\partial_{\vec{q}} K_{Berry}\big|_{q=0}$ and the Result of Bazaliy, Jones and Zhang

In contrast to the current free and the unpolarized current case, the Berry Phase kernel has now a linear $\vec{q}$ dependence.

$$
\begin{aligned}
& Re[K_{Berry}] = \frac{U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\, q\cos\chi \Big[\\
& \frac{1}{2\,a^2}\Big[\frac{3(\Delta'-a^2)^2}{a^2} - 4(a^2-\Delta') - (k_F^+)^2\Big]\,\log\Big|\frac{\Delta'-a^2-a\,k_F^+}{\Delta'-a^2+a\,k_F^+}\Big| + \frac{3\Delta k_F^+}{a^3} + \frac{k_F^+}{a} \\
& -\frac{1}{2\,a^2}\Big[\frac{3(\Delta'+a^2)^2}{a^2} - 4(\Delta'+a^2) - (k_F^-)^2\Big]\,\log\Big|\frac{\Delta'+a^2-a\,k_F^-}{\Delta'+a^2+a\,k_F^-}\Big| - \frac{3\Delta k_F^-}{a^3} + \frac{k_F^-}{a}\Big] \\
& \qquad \approx \frac{4\,\rho}{\Delta} a\cos\chi q = \frac{4\,\rho}{\Delta}\vec{a}\vec{q}
\end{aligned} \tag{6.11}
$$

With the spin wave stiffness ρ, as in the current free case, and the angle χ between $\vec{q}$ and $\vec{a}$.

If we Fourier transform the result in real space, the vector $\vec{q}$ becomes a gradient. Due to the functional derivative we obtain a modified Landau Lifschitz equation.

$$
\begin{aligned}
\dot{\vec{n}} &= \rho\nabla^2\vec{n} - 4\rho a_i \nabla_i \vec{n} \\
&= \rho\nabla^2\vec{n} - \frac{4\,m\,\rho}{\hbar\,n}\, j_i\,\nabla_i\vec{n}
\end{aligned} \tag{6.12}
$$

We have recovered the gradient term (1.2), previously derived by Bazaliy, Jones and Zhang.

By the use of the path integral formalism, we are able to establish the missing connection of the current introduced by Bazaliy Jones and Zhang to fix the units, and the physical current flowing through the ferromagnet.

This gradient term shifts the magnon dispersion downward, so Bazaliy Jones and Zhang assumed, that this generates the same sort of instability, as the anti-parallel magnetic field does (3.7). But the diagonal kernel K_π generates an additional mass term, which may stabilize the system. To confirm or refute their guess, we have to examine the spin wave gap.

6.2.2 $K_\pi\big|_{q=0}$ and the Spin Wave Gap

The calculation in Appendix C leads to:

$$
\begin{aligned}
&Re[K_\pi] = \frac{2U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\Big[\\
&\frac{2(k_F^-)^3}{3\Delta'} - \Big(\frac{(\Delta'+a^2)\,k_F^-}{a^2} + \frac{1}{2\,a}\Big(\frac{(\Delta'+a^2)^2}{a^2} - (k_F^-)^2\Big)\,\log\Big|\frac{\Delta'+a^2-a\,k_F^-}{\Delta'+a^2+a\,k_F^-}\Big|\Big)\\
&-\frac{2(k_F^+)^3}{3\Delta'} + \Big(\frac{(\Delta'-a^2)\,k_F^+}{a^2} + \frac{1}{2\,a}\Big(\frac{(\Delta'-a^2)^2}{a^2} - (k_F^+)^2\Big)\,\log\Big|\frac{\Delta'-a^2-a\,k_F^+}{\Delta'-a^2+a\,k_F^+}\Big|\Big)\Big]\\
&\quad\approx 4\frac{\rho}{\Delta}\vec{a}^2
\end{aligned}
\tag{6.13}
$$

The reappearance of the spin wave stiffness is very interesting. If we combine the current free case (2.56), which is also a second order term, the gradient term (C.9), and the gap, and calculate the dispersion relation (2.58), we find for a small current:

$$
\begin{aligned}
\Omega + 4\rho\vec{q}\vec{a} &= \rho\vec{q}^2 + 4\rho\vec{a}^2\\
\Rightarrow \Omega &= \rho(\vec{q}-2\vec{a})^2
\end{aligned}
\tag{6.14}
$$

The primary effect of a (small) current is just to shift the magnon dispersion relation by twice the wave vector $\vec{a}$.

This can be interpreted as if the collective modes got a Gallilei transformation from a resting frame of reference, to one, which moves with constant velocity.

Therefore we disproved Bazaliy Jones and Zhang's assumption of a spin wave instability due to the gradient term.

6.2.3 $\partial_\Omega K_{Berry}\big|_{q=0}$, $\partial^2_{\vec{q}} K_\pi\big|_{q=0}$ and the exact Magnon Dispersion relation

To plot the magnon dispersion relations, we need the two remaining coefficients. As you can see in figure C.4, the variation of $\partial^2_{\vec{q}} K_\pi\big|_{q=0}$ is very small with or without current. The most interesting aspect is, that the spin wave stiffness is slightly different for magnons parallel or perpendicular to the current direction.

Much more exciting is the energy proportionality factor $\partial_\Omega K_{Berry}\big|_{q=0}$.

The plot uses the units described and adopted in Appendix C. The prefactor diverges at the critical current density, so there the energy of creating magnons with arbitrary momentum vanishes. And this is independent of the magnon propagation direction

6.2.4 Current and the gap due to anisotropy

If the prefactor of the frequency diverges, what happens with a gap due to anisotropy? As a model we choose an external field, so we can use the claculations form Chapter 3. If we include the additional terms of the changed

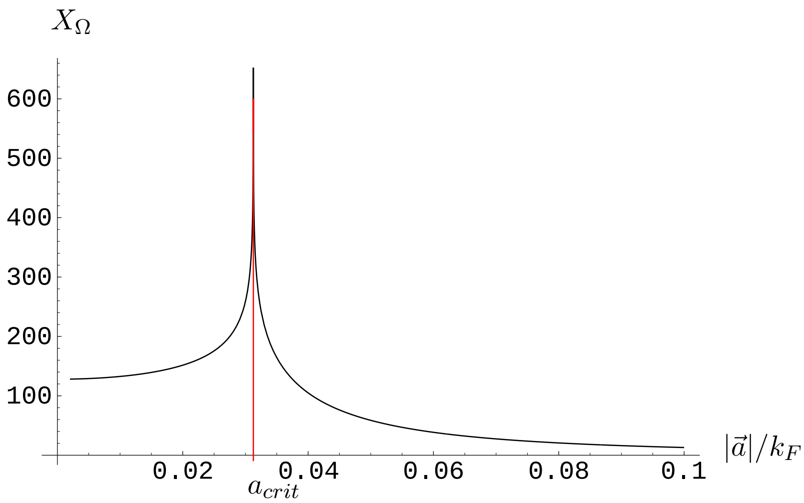


Figure 6.2: Proportionality factor of Ω

particle energy in equation 3.6, and Taylor develop in the gap instead of the energy, we get the same Taylor expansion coefficient:

$$Re[K_\pi]\Big|_{q=0} \approx \pm\,\partial_\Omega K_{Berry}\big|_{q=0} \cdot g\mu_B B \tag{6.15}$$

Therefore, if we divide by $\partial_\Omega K_{Berry}\big|_{q=0}$ to get the final dispersion relation, this coefficient cancels. So the current has no influence on the gap due an external field. Therefore this effect also generates no instability of the direction of magnetization.

6.2.5 Plots of selected magnon dispersion relations

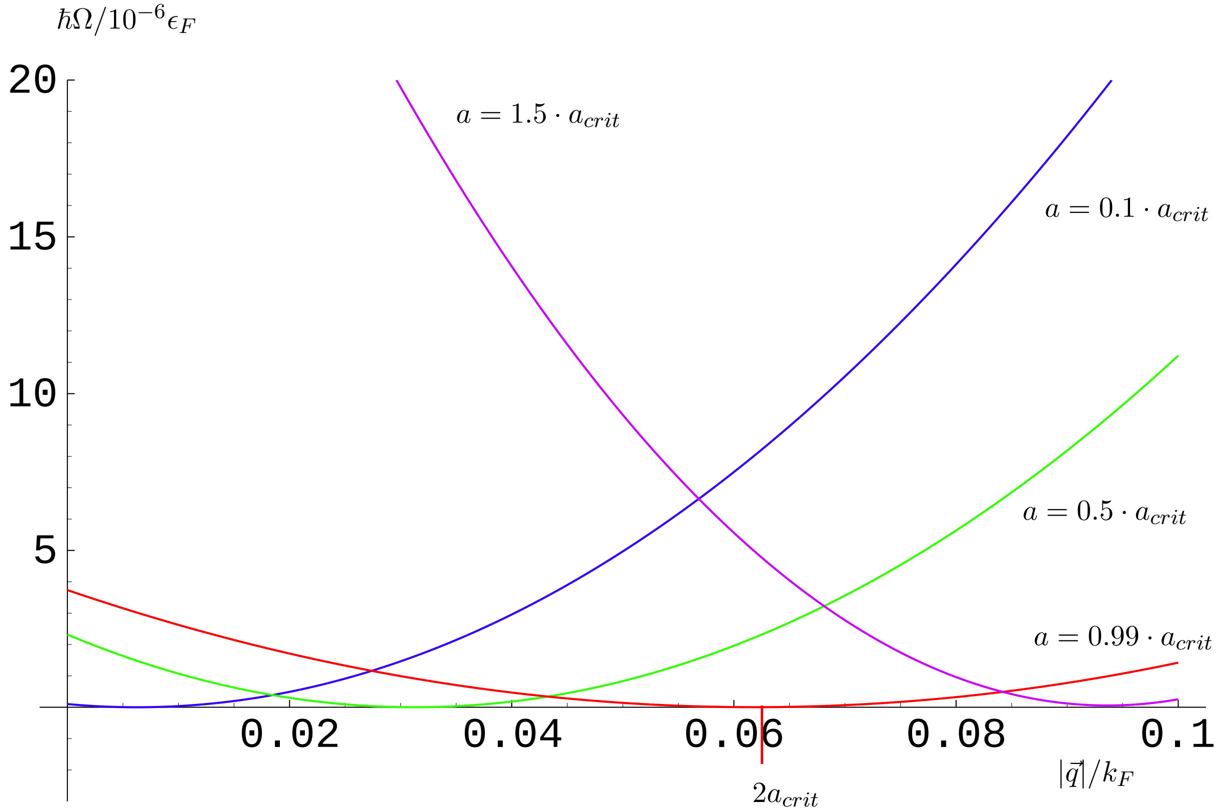


Figure 6.3: Dispersion relation for Magnons parallel to the current

Chapter 7

Inhomogeneous Spin-current and the MRAM writing

The next step to complicate the theory is to allow a space dependent but static current. Therefore the vector potential field $\vec{A}_{(\vec{p})}$ has no time dependence and so is a classical field. This choice is motivated by a phenomenological description of a decaying spin current due to various interactions.

This has a problematic aspect. This current configuration does not resemble a mean field solution. So the results of the calculation may be questionable, because we overexert this equilibrium formalism to a non equilibrium situation. See this as a general warning.

Because the resulting Green's function (5.5) is not longer diagonal in $\vec{k}$, we have to include the current via perturbation theory. We derive the homogeneous mean field solution for the magnetic field without a current, just like equation 2.20, and choose the direction of the magnetization parallel to the polarization, so $\vec{n}_c = \vec{n}$. Then we include fluctuations in magnetization and the current interaction perturbatively.

$$G^{-1} = G_o^{-1} - \frac{U}{3}\delta\vec{h}_q\vec{\tau} + (\vec{v}_p\vec{A}_{(\vec{p})})\vec{n}_c\vec{\tau} \quad \equiv \quad G_o^{-1} + V_h + V_A \tag{7.1}$$

Using the mathematical modus operandi described in the second Chapter, we get:

$$
\begin{aligned}
S_{(\vec{h},\delta\vec{h})} &= S_{o(\vec{h})} \\
&+ \frac{1}{N\beta}\vec{h}\delta\vec{h}_{(q=0)} - \mathrm{Tr}[G_o\, V_h] \\
&+ \sum_{j=1}^{\infty}(-1)^{j+1}\cdot \mathrm{Tr}[G_o\, V_h (G_o\, V_A)^j] \\
&+ \frac{1}{N\beta}\sum_q \frac{\delta\vec{h}_{(q)}\,\delta\vec{h}_{(-q)}}{2} + \mathrm{Tr}[G_o\, V_h\, G_o\, V_h] \\
&+ \mathrm{Tr}[G_o\, V_A\, G_o\, V_A] \\
&+ O(V_A \cdot V_h^2)
\end{aligned}
\tag{7.2}
$$

The first term is the unperturbated action, the second line cancels, as already shown in Chapter 2. The fourth line produces the usual Spin wave excitations, and the fifth line generates the Kubo formula for conduction, if we functional derive with respect to $\vec{A}_{(\vec{q}_o)}$.

But the third line is crucial. Now we have a term linear in $\delta\vec{h}$, so our mean field solution calculated for the current free case seems to be not the right ground state any more.

7.1 Vanishing first order in $\vec{v}_q\vec{A}_{(q)}$

Due to the spin trace this disturbing linear term linear in $\delta\vec{h}$ can be eliminated. Because of our choice, to align the magnetization and polarization direction parallel, the result of the spin trace is a function of one vector $\delta\vec{h}$, and a varying number of vectors $\vec{n} = \vec{n}_c$. If we examine, how the trace combines vectors in Appendix A, there are only two possible ways. One is due to an epsilon tensor, but because only one vector differs from all the others, all tensor products must vanish $\vec{n}(\delta\vec{h} \times \vec{n}) = 0$.

The other contribution arises from the scalar products $\delta\vec{h} \cdot \vec{n} = \sigma$, and therefore the linear term is a function of only the longitudinal fluctuations.

$$\mathrm{Tr}[G_o V_h (G_o V_A)^j] = F_j[\delta\vec{h} \cdot \vec{n}] = F_j[\sigma] \neq F_j[\sigma, \pi] \tag{7.3}$$

But the longitudinal fluctuations are massive excitations, and if the ferromagnet is hard enough, we can neglect these excitation. Then, all terms first order in $\delta\vec{h} \doteq \vec{\pi}$ vanish, and our mean field calculation for the magnetization really extremizes the action.

7.2 Lowest Order Current Spin-wave Interaction

The lowest order contribution with an interaction is the third order.

$$\mathrm{Tr}[(G_o V_h + G_o V_A)^3] = \mathrm{Tr}[(G_o V_h)^3] + \mathrm{Tr}[(G_o V_A)^3] + 3 \cdot \mathrm{Tr}[G_o V_h G_o V_h G_o V_A] \tag{7.4}$$

All the terms linear in $\delta\vec{h}$ can be neglected as discussed in the previous section. We also can neglect the term with three current interactions, because this is only the quadratic correction of the conductivity. The same argument also holds for the three magnon interaction. All whats left is a current-magnon-magnon bubble.

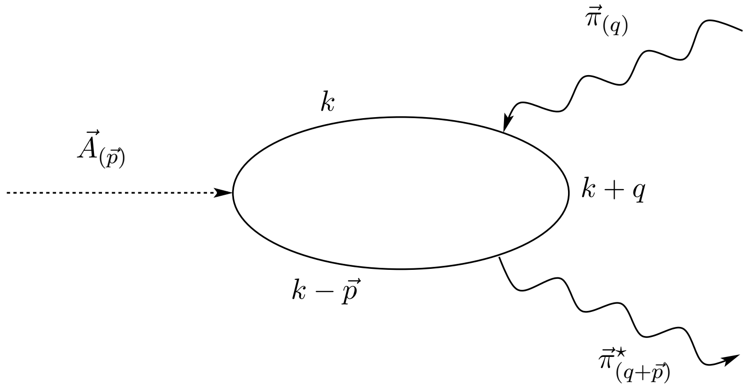


Figure 7.1: Current Magnon Magnon Bubble

With $g_o = g_1 + g_\tau \vec{n}\vec{\tau}$, and the usual definition of g_1 and g_τ (2.18), we evaluate the spin trace

$$\begin{aligned}\mathrm{Tr}[G_o V_h G_o V_h G_o V_A] &= \frac{1}{(V\beta)^3}\sum_p (\vec{v}_{(p)}\vec{A}_{(p)}) \cdot \Big[\\ \sum_q i\vec{\pi}^\star_{(q)} \cdot (\vec{\pi}_{(q+p)} \times \vec{n}) &\sum_k g_{-(k)}g_{+(k+q)}g_{-(k-p)} + g_{+(k)}g_{-(k+q)}g_{+(k-p)} \\ + \sum_q \vec{\pi}^\star_{(q)} \cdot \vec{\pi}_{(q+p)} &\sum_k g_{-(k)}g_{+(k+q)}g_{-(k-p)} - g_{+(k)}g_{-(k+q)}g_{+(k-p)} \Big]\end{aligned} \tag{7.5}$$

We can define a Berry phase like kernel L_{Berry} and a diagonal kernel L_π analog to the previous cases.

$$\mathrm{Tr} = \frac{1}{(V\beta)^2}\sum_{p\,q} i\vec{\pi}^\star_{(q)} \cdot (\vec{\pi}_{(q+p)} \times \vec{n})\, L_{Berry} + \vec{\pi}^\star_{(q)} \cdot \vec{\pi}_{(q+p)}\, L_\pi \tag{7.6}$$

with

$$\begin{aligned}L_{Berry} &= \frac{1}{(V\beta)}\sum_k [g_{-(k)}g_{+(k+q)}g_{-(k-\vec{p})} + g_{+(k)}g_{-(k+q)}g_{+(k-\vec{p})}](\vec{k}-\vec{p}/2)\vec{A}_{(\vec{p})} \\ L_\pi &= \frac{1}{(V\beta)}\sum_k [g_{-(k)}g_{+(k+q)}g_{-(k-\vec{p})} - g_{+(k)}g_{-(k+q)}g_{+(k-\vec{p})}](\vec{k}-\vec{p}/2)\vec{A}_{(\vec{p})}\end{aligned}$$

7.3 Calculation of the Kernels L_π and L_{Berry}

As usual, the first step of calculating the kernels is the evaluation of the frequency sum $\omega_n = k^o$

$$\sum_{k^o} g_{-(k)} g_{+(k+q)} g_{-(k-\vec{p})} = \frac{n_F(E^+_{\vec{k}+\vec{q}})}{E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}} + \Omega_n} \frac{1}{E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}-\vec{p}} + \Omega_n}$$
$$+ \Big[\frac{n_F(E^-_{\vec{k}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n} - \frac{n_F(E^-_{\vec{k}-\vec{p}})}{E^-_{\vec{k}-\vec{p}} - E^+_{\vec{k}+\vec{q}} - \Omega_n} \Big] \frac{1}{E^-_{\vec{k}} - E^-_{\vec{k}-\vec{p}}} \tag{7.7}$$

The other sum can be generated form tis result by exchanging the upper (+) and (−) indices.

The described decay of the spin current should happen over a much longer distance compared to the lattice constant. Therefore $\vec{p}$ will be much smaller than the typical momentum of an electron, and we can develop the resulting kernels in $\vec{p}$. Unfortunately, at $\vec{p} = 0$, the result 7.7 becomes singular, which leads in the right limit to a derivative. To expand this object, we use:

$$\frac{f(x) - f(x-\epsilon)}{\epsilon} = \frac{f(x) - \big[f(x) - f'(x)\epsilon + \frac{1}{2} f'(x)\epsilon^2 + ...\big]}{\epsilon}$$
$$= f'(x) - \frac{1}{2} f'(x)\epsilon + O(\epsilon^2) \tag{7.8}$$

with $x = E_{\vec{k}}$ and $\epsilon = E^\pm_{\vec{k}} - E^\pm_{\vec{k}-\vec{p}} = \frac{\hbar^2}{m}(\vec{k}\vec{p} - \frac{p^2}{2})$ we get the result up to linear order in $\vec{p}$:

$$\sum_{k^o} g_{-(k)}g_{+(k+q)}g_{-(k-\vec{p})} = \frac{n_F(E^+_{\vec{k}+\vec{q}})}{E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}} + \Omega_n}\,\frac{1}{E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}} + \Omega_n} \tag{7.9}$$
$$+\frac{n_F(E^+_{\vec{k}+\vec{q}})}{E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}} + \Omega_n}\,\frac{1}{(E^+_{\vec{k}+\vec{q}} - E^-_{\vec{k}} + \Omega_n)^2}\,\frac{\hbar^2}{m}(-\vec{k}\vec{p})$$
$$+\frac{n'_F(E^-_{\vec{k}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n} - \frac{n_F(E^-_{\vec{k}})}{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n)^2}$$
$$-\frac{1}{2}\Big[\frac{n''_F(E^-_{\vec{k}})}{E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n} - 2\frac{n'_F(E^-_{\vec{k}})}{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n)^2} + 2\frac{n_F(E^-_{\vec{k}})}{(E^-_{\vec{k}} - E^+_{\vec{k}+\vec{q}} - \Omega_n)^3}\Big]\frac{\hbar^2}{m}(\vec{k}\vec{p})$$

Now we have to consider, which order is important. In Chapter 2 we developed a gausian theory in $\vec{\pi}$ up to second order in Ω and $\vec{q}$. In the homogeneous current case we learned, that $\vec{a} = \hbar\vec{A}$ is of same magnitude as $\vec{q}$, so we calculated everything up to second order but every parameter $\vec{q}$, $\vec{A}$ and Ω increases the order by one.

Now we have the additional parameter $\vec{p}$, which is also small. To not exceed the second order in all four parameters, we keep only linear contributions, because we are already linear in $\vec{A}$. But a development in something else than $\vec{p}$ is equivalent to the case of a homogeneous current. The only new term can arise from the linear term in $\vec{A}$ and $\vec{p}$, and therefore we set Ω and $\vec{q}$ equal to zero. Then the frequency sums simplify:

$$\sum_{k^o} g_{-(k)}g_{+(k+q)}g_{-(k-\vec{p})}\cdot(\vec{k}-\vec{q}/2)\vec{A} = \tag{7.10}$$
$$-\frac{n_F(E^+_{\vec{k}})}{2\Delta^2}\,p\,A + \frac{n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}})}{\Delta^3}\,k^2\,p\,A\,\cos^2[\theta]$$
$$-\Big(\frac{n'_F(E^-_{\vec{k}})}{2\Delta} + \frac{n''_F(E^-_{\vec{k}})}{2\Delta}\frac{\hbar^2\,k^2}{m}\cos^2[\theta] + \frac{n_F(E^-_{\vec{k}})}{2\Delta^2} + \frac{n'_F(E^-_{\vec{k}})}{\Delta^2}\frac{\hbar^2\,k^2}{m}\cos^2[\theta]\Big)\,p\,A$$

The expression in the bracket can be simplified by transforming the derivation of the distribution with respect to energy to a derivation with respect to k and a partial integration:

$$\int_{-1}^{1} d\cos[\theta] \int_0^\infty dk\, k^2 \,\frac{n''_F(E^-_{\vec{k}})}{2\Delta} \frac{\hbar^2\, k^2}{m} \,\cos^2[\theta] = \frac{2}{3}\int_0^\infty dk\, \frac{n''_F(E^-_{\vec{k}})}{2\Delta}\frac{\hbar^2\, k^4}{m}$$
$$= \frac{2}{3}\int_0^\infty dk\, \frac{\partial_k n'_F(E^-_{\vec{k}})}{2\Delta}\frac{1}{\frac{\hbar^2\, k}{m}}\frac{\hbar^2\, k^4}{m} = -\frac{2}{3}\int_0^\infty dk\, \frac{n'_F(E^-_{\vec{k}})}{2\Delta}\partial_k\, k^3$$
$$= -\int_{-1}^{1} d\cos[\theta] \int_0^\infty dk\, k^2 \,\frac{n'_F(E^-_{\vec{k}})}{2\Delta} \tag{7.11}$$

and completely analog

$$\sum_{\vec{k}} \frac{n'_F(E^-_{\vec{k}})}{\Delta^2}\frac{\hbar^2\, k^2}{m}\,\cos^2[\theta] = -\sum_{\vec{k}}\frac{n_F(E^-_{\vec{k}})}{\Delta^2}$$

So the final result for the frequency sum is

$$\sum_{k^o} g_{-(k)}g_{+(k+q)}g_{-(k-\vec{p})}\cdot\frac{\hbar^2(\vec{k}-\vec{p}/2)\vec{a}}{m} =$$
$$-[\frac{n_F(E^+_{\vec{k}})+n_F(E^-_{\vec{k}})}{\Delta^2} + 4\frac{n_F(E^+_{\vec{k}})-n_F(E^-_{\vec{k}})}{\Delta^3}\frac{\hbar^2(k\cos\theta)^2}{2m}]\frac{\hbar^2\vec{p}\,\vec{a}}{2m} \tag{7.12}$$

whith equation C.4 we replaced $(\vec{k}\vec{A})(\vec{k}\vec{p}) = k^2\cos\theta\vec{p}\vec{a}$. This is possible, because the additional missing term is linear in $\sin\phi$ and vanishes due to the angular integration.

The result for the other frequency sum can be generated by replacing the upper indices $(+)$ and $(-)$, and changing the sign of Δ, which is exactly the same. Therefore, the kernels are

$$
\begin{aligned}
L_\pi &= 0 \\
L_{Berry} &= \frac{-2U}{3N} \\
&\qquad \sum_{\vec{k}} \Big[\frac{n_F(E^+_{\vec{k}}) + n_F(E^-_{\vec{k}})}{\Delta^2} + 4 \frac{n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}})}{\Delta^3} \, \frac{\hbar^2 \, (k \cos\theta)^2}{2m} \Big] \frac{\hbar^2 \vec{p}\,\vec{a}}{2m} \\
&= -\frac{\rho}{\Delta} \, \vec{p}\vec{A}_{(\vec{p})} \qquad (7.13)
\end{aligned}
$$

The numerical prefactor is again the spin wave stiffness ρ as defined in 2.56.

7.4 Results

The modified effective action for the bosonic vector field $\vec{pi}$ now looks like:

$$
\begin{aligned}
S^{(2)}_{\vec{\pi}} \;&=\; \frac{1}{2N\beta\Delta} \sum_q \; \rho \vec{q}^{\,2} \, \vec{\pi}_{(-q)} \vec{\pi}_{(q)} \;+\; i\Omega \, \vec{\pi}_{(-q)} \big(\vec{n} \times \vec{\pi}_{(q)} \big) \\
&\qquad + i\rho \, (\vec{p}\vec{A}_{(\vec{p})}) \;\; \vec{\pi}_{(-q)} \big(\vec{n} \times \vec{\pi}_{(\vec{p}+q)} \big) \qquad (7.14)
\end{aligned}
$$

If we Fourier transform the expression back in real space, we get:

$$
\dot{\vec{\pi}}_{(\vec{x},t)} = +\rho \Delta \vec{\pi}_{(\vec{x},t)} \;\times\; \vec{n} + \rho \nabla \vec{I}_{(\vec{x})} \, (\vec{n} \times \vec{\pi}_{(\vec{x},t)}) \times \vec{n} \qquad (7.15)
$$

This is a local version of the Slonczewski term. To reproduce his original expression, we average over space, so the divergence of the current is just the incoming minus the outgoing.

$$
\dot{\vec{\pi}}_{(t)} = +\rho \Delta \vec{\pi}_{(t)} \;\times\; \vec{n} + \rho \big(\vec{I}_{in} - \vec{I}_{out} \big) \, (\vec{n} \times \vec{\pi}_{(t)}) \times \vec{n} \qquad (7.16)
$$

Slonczewski also derived an additional prefactor which mainly results from the different conductivity of the spin parallel and anti-parallel configuration. This effect is here not included, because $\vec{I}$ is the enforced current inside of the magnet.

Chapter 8

Conclusions

In this work we have discussed a model to describe currents crossing a ferromagnetic conductor, and their effects on the magnetization.

We enforced a constraint to the partition function which looks like Lagrange multiplier. We interpreted the multiplier as a physical vector potential $\vec{A} = \frac{\hbar}{e}\vec{a}$, which generates a current. The further discussion of the model revealed several results.

8.1 Two Kinds of Currents

We have seen, we can enforce two different kinds of current. By enforcing a current with a given amperage, we get a change of the electron dispersion relation as shown in figure 8.1a. The system chooses an appropriate spin structure for the current, and therefore this kind of current does not interact with the magnetization at all.

Then we enforced only a spin current, and let the system choose the amperage of the resulting electrical current. This generates the electron dispersion relation 8.1b. We have shown in Section 5.3, that this two kinds of currents superpose linearly, so by a proper addition of both cases, a physical

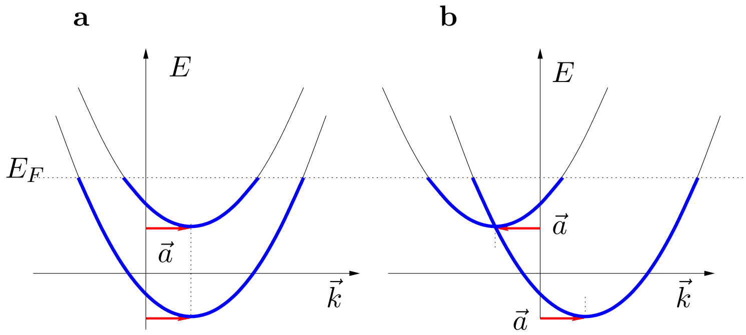


Figure 8.1: Shifted electron bands due to different vector potentials

current with independent amperage and spin structure can be described.

Remains the question, if such a current of type 2 has any physical meaning. Inside a bulk material, this current will decay over the spin diffusion length. So the homogeneous spin current used in the calculation can be seem as approximation for the first layer of the magnet, up to a depth of the spin diffusion length.

On the other hand, it is known, that the mean free path of spin up and donw electrons can vary [23], and the conductivity of transition metals becomes spin dependent [24]. Therefore, we get a homogeneous current of type 2, which strength is determined by the difference of the spin up and down conductivity.

8.2 Enhanced Dissipation

When the spin-current density is high enough, the Fermi spheres are shifted against each other and at a "critical" current density a_{crit}, they touch:

$$k_F^+ + a_{crit} = k_F^- - a_{crit} \tag{8.1}$$

The shift due to the current closes the Zeeman splitting gap, and the system starts to absorb and emit magnons, which have a diminutive energy compared to the degenerate electron gas.

For typical values of Cobalt ($\mu \approx 7eV$, $n = 9 \cdot 10^{22} \frac{1}{cm^3}$, $\Delta \approx 1eV$), we obtain a critical spin-current density of $8 \cdot 10^{10} \frac{A}{cm^2}$. This is not the physical needed current density, it is the needed variation of the spin-current, which is already created by applying a current to a ferro-magent with splitted bands. Nevertheless it is approximately of the same magnitude, and so about two magnitudes too big to explain the abrupt rise in the resistivity, measured by Tsoi et al. [2] and E.B.Myers et al. [1]. It is consequently unlikely, that the resistivity step is related.

But this effect may be visible in ferromagnetic semiconductors, which have a much smaller Zeeman splitting and therefore need a much smaller current density.

8.3 Shift of the Dispersion Relation

The spin polarized current also changes the spin wave dispersion relation. We include the homogeneous current exactly, and obtain additional terms. In a weak current approximation, one term is linear in the magnon momentum $\vec{q}$, which reproduces the result Bazaliy, Jones and Zhang [3] due to Fourier transformation and functional derivation:

$$\dot{\vec{n}} = \rho\nabla^2\vec{n} - \frac{4\,m\,\rho}{\hbar\,n}\,j_i\,\nabla_i\vec{n} \tag{8.2}$$

with the spin wave stiffness ρ. We even advance their result by determing the missing connection of the current density j and the physical current going through the system.

But we refute their conclusion, this would cause a spin wave instability. In their discussion, they are not consistent in the order of approximation, so they missed a stabilizing spin wave gap of the same order. Instead of causing an instability, the magnon dispersion relation is shifted by the wave vector of the vector potential.

$$\omega = \rho(\vec{k} - \vec{a})^2 \tag{8.3}$$

This effect can be interpreted as translating the collective spin waves in a moving frame of reference. The variation of the magnetization gets washed away by the current flow.

The spin wave stiffness itself is also affected by the current. At the critical current, where the two Fermi spheres touch the stiffness is vanishing, so the energy to excite spin waves with arbitrary momentum vanishes, as you can see in figure 6.3 (page 56). But a spin wave gap due external fields (as model for anisotropy) is not affected by the effect.

8.4 Slonczewski's torque on the magnetization

We modeled a decaying spin current of type 2, which we treated via perturbation expansion. In third order, first in current, second in magnons, we reproduce Slonczewski's results. We derive the local version of his addend to the Landau Lifschitz equation:

$$\dot{\vec{\pi}}_{(\vec{x},t)} = +\rho\Delta\vec{\pi}_{(\vec{x},t)} \times \vec{n} + \rho\nabla\vec{I}_{(\vec{x})} \left(\vec{n} \times \vec{\pi}_{(\vec{x},t)}\right) \times \vec{n} \tag{8.4}$$

The divergence of the spin current acts like a source of angular momentum, which approves Slonczewski's interpretation of the effect. We now also see the connection between this torque term and the result of Bazaliy, Jones and Zhang [3]. They arise from the same current magnetization interaction, but they represent different limits.

8.5 Open questions

We still have not completed the discussion of the dissipative term corresponding to the term generating Slonczewski's torque. This may leading to the spin wave laser instability for spin waves (swaser), proposed by Berger [8]. And another open question is,how more realistic assumptions effects this toy model calculation, especially a non quadratic dispersion relation and disorder.

Appendices

Appendix A

Traces of Pauli Matrices

To calculate traces of Pauli matrices we use the product rule, which represents the anti-commutation properties and the Lee algebra.

$$\tau^a \tau^b = \delta^{ab} + i\epsilon^{abc}\tau^c \tag{A.1}$$

With this we can reduce a product of Pauli matrices by one matrix. Therefore we start with the trace of one Pauli matrix and increase the number by one

1. **One Pauli Matrix** (Without prove)

$$\mathrm{Tr}[\tau^a] = 0 \tag{A.2}$$

2. **Two Pauli Matrices:** Due to the product rule, we can reduce the trace of two Pauli matrices to the trace of one.

$$\mathrm{Tr}[\tau^a \tau^b] = \mathrm{Tr}[\delta^{ab} + i\epsilon^{abc}\tau^c] = 2\delta^{ab} \tag{A.3}$$

3. **Three Pauli Matrices:** The trace of three can calculated with the trace of two Pauli matrices:

$$\mathrm{Tr}[\tau^a \tau^b \tau^c] = \mathrm{Tr}[\tau^a(\delta^{bc} + i\epsilon^{bcd}\tau^d] = 2i\delta^{ad}\epsilon^{bcd} = 2i\epsilon^{abc} \tag{A.4}$$

4. **Four Pauli Matrices:** And so on.

$$
\begin{aligned}
\mathrm{Tr}[\tau^a\tau^b\tau^c\tau^d] &= \mathrm{Tr}[\tau^a\tau^b(\delta^{cd} + i\epsilon^{cde}\tau^e] \\
&= 2\delta^{ab}\delta^{cd} - 2\epsilon^{abe}\epsilon^{cde} \\
&= 2\delta^{ab}\delta^{cd} - 2\delta^{ac}\delta^{bd} + 2\delta^{ad}\delta^{bc} \qquad \text{(A.5)}
\end{aligned}
$$

5. **Five Pauli Matrices**:

$$
\begin{aligned}
\mathrm{Tr}[\tau^a\tau^b\tau^c\tau^d\tau^e] &= \mathrm{Tr}[\tau^a\tau^b\tau^c(\delta^{de} + i\epsilon^{def}\tau^f] \\
&= 2i\epsilon^{abc}\delta^{de} + 2i\epsilon^{def}(\delta^{ab}\delta^{cf} - 2\delta^{ac}\delta^{bf} + 2\delta^{af}\delta^{bc}) \\
&= 2i\delta^{de}\epsilon^{abc} + 2i\delta^{ab}\epsilon^{dec} - 2i\delta^{ac}\epsilon^{deb} + 2i\delta^{bc}\epsilon^{dea} \quad \text{(A.6)}
\end{aligned}
$$

6. **Six Pauli Matrices**:

$$
\begin{aligned}
\mathrm{Tr}[\tau^a\tau^b\tau^c\tau^d\tau^e\tau^f] = \; & +2\delta^{ab}\delta^{cd}\delta^{ef} - 2\delta^{ab}\delta^{ce}\delta^{df} + 2\delta^{ab}\delta^{cf}\delta^{de} \\
& -2\delta^{ac}\delta^{bd}\delta^{ef} + 2\delta^{ac}\delta^{be}\delta^{df} - 2\delta^{ac}\delta^{bf}\delta^{ed} \\
& +2\delta^{ad}\delta^{bc}\delta^{ef} - 2\delta^{ad}\delta^{be}\delta^{cf} + 2\delta^{ad}\delta^{bf}\delta^{ce} \\
& -2\delta^{ae}\delta^{bf}\delta^{cd} + 2\delta^{ae}\delta^{bd}\delta^{cf} - 2\delta^{ae}\delta^{bc}\delta^{df} \\
& +2\delta^{af}\delta^{cd}\delta^{be} - 2\delta^{af}\delta^{bd}\delta^{ce} + 2\delta^{af}\delta^{bc}\delta^{de} \qquad \text{(A.7)}
\end{aligned}
$$

Every higher order of traces can be successive solved through the system. But the algebraic complexity rises strongly.

Appendix B

Calculation of the Dissipation Sum

This Appendix is dedicated to the calculation of the two sums, which compose the diagonal kernel (2.50) and Berry phase kernel (2.44). Because the impulse of a spin wave is always neglectable compared to electrons, we evaluate them in the $\vec{q} = 0$ limit.

B.1 $\widetilde{E}^{-}_{\vec{k}} - \widetilde{E}^{+}_{\vec{k}} + \hbar\Omega = 0$

We start with the sum, which describes the absorption of a magnon due to flipping of a majority spin into a minority spin.

$$\begin{aligned} & \sum_{\vec{k}} \left[n_F(\widetilde{E}^{-}_{\vec{k}}) - n_F(\widetilde{E}^{+}_{\vec{k}}) \right] \delta_{(\widetilde{E}^{-}_{\vec{k}} - \widetilde{E}^{+}_{\vec{k}} + \hbar\Omega)} \\ = & \sum_{\vec{k}} \left[n_F(E^{-}_{\vec{k}-\vec{a}}) - n_F(E^{+}_{\vec{k}+\vec{a}}) \right] \delta_{(-\frac{2\hbar^2 \vec{k}\vec{a}}{m} - \Delta + \hbar\Omega)} \\ = & \sum_{\vec{k}} \left[n_F(E^{-}_{\vec{k}}) - n_F(E^{+}_{\vec{k}+2\vec{a}}) \right] \delta_{(-2\frac{\hbar^2 \vec{k}\vec{a}}{m} - 2\frac{\hbar^2 a^2}{m} - \Delta + \hbar\Omega)} \end{aligned} \tag{B.1}$$

In the second line, we neglected the shift in the chemical potential due to gauge invariance, and in the third line we shifted the summation index by $\vec{a}$.

To use the δ-function properly, we transform the sum in an integral.

$$\sum_k = \frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos[\theta] \int_0^{\infty} dk k^2 \; \Big[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+\vec{A}}) \Big] \cdot \\ \cdot \delta_{\left(-\frac{2\hbar^2 ka}{m}\cos\theta - \frac{2\hbar^2 a^2}{m} - \Delta + \hbar\Omega\right)} \tag{B.2}$$

We can simplify the expression by integrating over ϕ and dividing the δ-function entree by $2\hbar^2 ak/m$, while defining $\Omega' = \frac{m\Omega}{2\hbar}$ and $\Delta' = \frac{m\Delta}{2\hbar^2}$.

$$\sum_k = \frac{V}{(2\pi)^2} \int_{-1}^{1} d\cos[\theta] \int_0^{\infty} dk k^2 \; \Big[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+2\vec{a}}) \Big] \cdot \\ \cdot \frac{m}{2\hbar^2 k|a|} \delta_{\left(\cos\theta + \frac{a}{k} + \frac{\Delta'}{ak} - \frac{\Omega'}{ak}\right)} \tag{B.3}$$

To solve this integral, we have to interpret it graphically. The δ-function defines a plane (*blue*) in the k-space, satisfying $k\cos\theta = -\big(a + \frac{\Delta'}{a} - \frac{\Omega'}{a}\big)$, as you can see in figure B.1. The result of the integral is the area, which lies inside the Fermi sphere, defined by $n_F(E^-)$, but outside of the Fermi sphere of $n_F(E^+)$. Now we have to distinguish several cases.

1. $k_F^- < a + \frac{\Delta'}{a} - \frac{\Omega'}{a}$

   If the current and therefore $\vec{a}$ is very small, the plane does not intersect at all with the Fermi sphere, so the integral is zero.

2. $k_F^- > a + \frac{\Delta'}{a} - \frac{\Omega'}{a} > k_F^+ + 2a$

   Now the plane intersect with the majority Fermi sphere, but not with the minority, as you can see in figure B.1B. The figures B.1A and B.1C show just the borders of this interval.

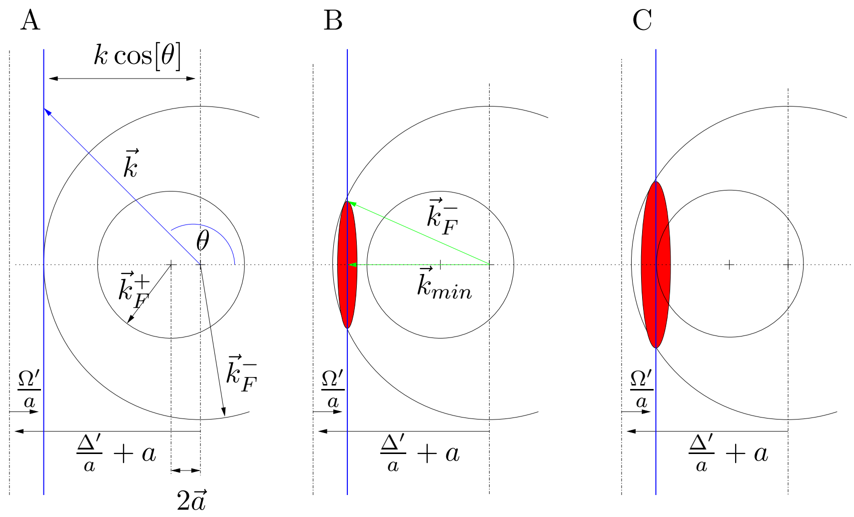


Figure B.1: Interval where only one Fermi sphere intersect the plane

As long as the minority sphere is not intersected, the Fermi distribution of E_k^+ is always zero. We use the other distribution as a upper limit, and the delta function to remove the angle integration. Then the momentum integration ranges from the distance of the plane to the Fermi momentum:

$$\sum_k = \frac{V}{(2\pi)^2}\frac{m}{2\hbar^2 a}\int_{k_{min}}^{k_F^-} dkk \tag{B.4}$$

$$= \frac{mV}{16\pi^2\hbar^2|a|}\Big[\,(k_F^-)^2 - \big(\ a + \frac{\Delta'}{a} - \frac{\Omega'}{a}\ \big)^2\Big] \tag{B.5}$$

3. $k_F^+ + 2a > a + \frac{\Delta'}{a} - \frac{\Omega'}{a} > 2a - k_F^+$

Now the plane intersects both Fermi spheres, as shown in figure B.2A-C. The area we have to integrate is the shaded ring. We can use the previous calculation up to (B.4). But now, the lower border of the integral is not

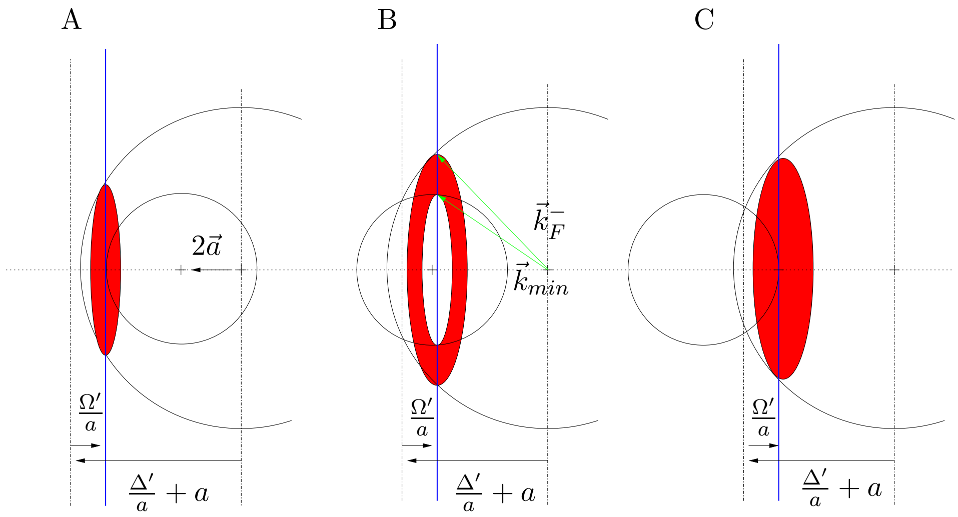


Figure B.2: Interval where both Fermi spheres intersect with plane

the distance to the plane any more. We can derive the new lower border of the integral with the figure B.3.

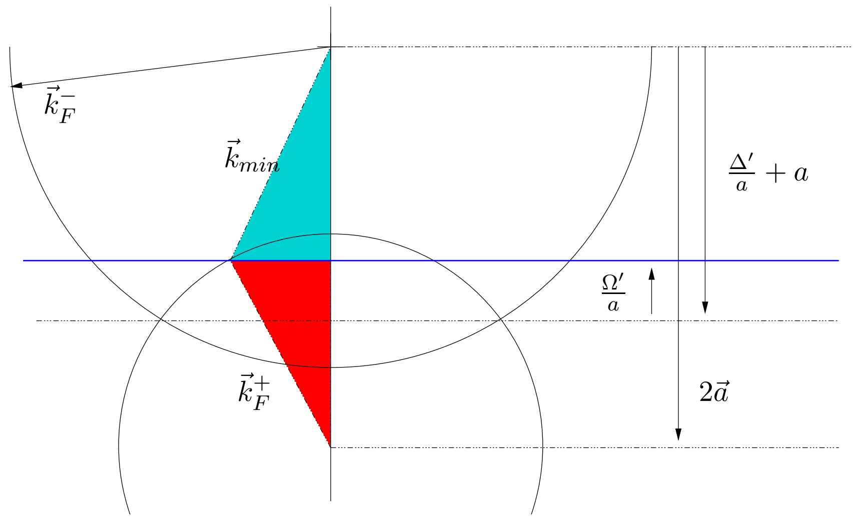


Figure B.3: Drawing to derive k_{min}

There are two rectangular triangles, with a common, but unknown side. By using the Phytagoraean theorem in both triangles we can write:

$$(k_F^+)^2 - \big(2a - (a + \frac{\Delta'}{a} - \frac{\Omega'}{a})\big)^2 = (k_{min})^2 - (a + \frac{\Delta'}{a} - \frac{\Omega'}{a})^2$$
$$\rightarrow (k_{min})^2 = (k_F^+)^2 + 4\Delta' - 4\Omega' \qquad \text{(B.6)}$$

To simplify this further, we have look, what k_F^+ is:

$$E^{\pm}_{(k_F^{\pm})} = 0 \qquad \rightarrow \qquad \frac{\hbar^2 (k_F^{\pm})^2}{2m} = \mu \mp \Delta/2$$
$$\rightarrow (k_F^{\pm})^2 = \frac{2m\mu}{\hbar^2} \mp \frac{m\Delta}{\hbar^2} = \frac{2m\mu}{\hbar^2} \mp 2\Delta' \qquad \text{(B.7)}$$

With this equation, we can replace $k_F^+ + 4\Delta'$ by k_F^-, and the integral (B.4) becomes:

$$\sum_k = \frac{V}{(2\pi)^2} \frac{m}{2\hbar^2 |a|} \int_{k_{min}}^{k_F^-} dk k = \frac{mV}{4\pi^2 \hbar^2 |a|} \Omega' \qquad \text{(B.8)}$$

4. $2a - k_F^+$ and $k_F^- > a + \frac{\Delta'}{a} - \frac{\Omega'}{a}$

   After the minority spin Fermi sphere passed through the relevant plane, we end with the same type of integral as in the second case.

$$\sum_k = \frac{V}{(2\pi)^2} \frac{m}{2\hbar^2 |a|} \int_{k_{min}}^{k_F^-} dk k \qquad \text{(B.9)}$$
$$= \frac{mV}{16\pi^2 \hbar^2 |a|} \Big[(k_F^-)^2 - \big(a + \frac{\Delta'}{a} - \frac{\Omega'}{a} \big)^2 \Big] \qquad \text{(B.10)}$$

5. $k_F^- < a\frac{\Delta'}{a} - \frac{\Omega'}{a}$

   Now, the relevant plane has left both Fermi spheres, so the sum is zero and therefore the dissipation is also zero.

We can draw the mathematical result as a function of Ω in arbitrary units.

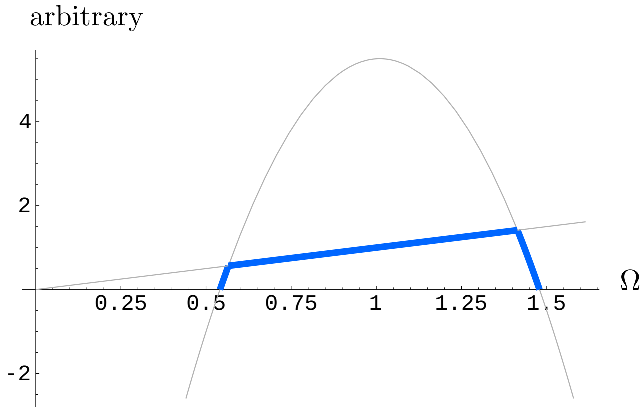


Figure B.4: Mathematical result in arbitrary units

This result has nearly the same structure as dissipation in a ground state Fermi system [25]. But one thing is very different. For very small $\vec{a}$'s, we have a gap. Magnons with less than a critical energy can not be absorbed by the Fermi system.

In the limit $\Omega \to 0$, we can neglect the on and off switching, because the width of this areas scale linear with Ω, and consider only the middle part, which is linear in Ω.

B.2 $\widetilde{E}^+_{\vec{k}} - \widetilde{E}^-_{\vec{k}} + \hbar\Omega = 0$

The calculation of this sum follows the same scheme.

$$
\begin{aligned}
& \sum_{\vec{k}} \Big[n_F(\widetilde{E}^+_{\vec{k}}) - n_F(\widetilde{E}^-_{\vec{k}}) \Big] \delta_{(\widetilde{E}^+_{\vec{k}} - \widetilde{E}^-_{\vec{k}} + \hbar\Omega)} \\
= & \sum_{\vec{k}} \Big[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}-2\vec{a}}) \Big] \delta_{(2\frac{\hbar^2 \vec{k}\vec{a}}{m} - 2\frac{\hbar^2 a^2}{m} + \Delta + \hbar\Omega)} \\
= & \frac{V}{(2\pi)^2} \int_{-1}^{1} d\cos[\theta] \int_0^\infty dk k^2 \, \Big[n_F(E^-_{\vec{k}}) - n_F(E^+_{\vec{k}+2\vec{a}}) \Big] \cdot \\
& \cdot \frac{m}{2\hbar^2 k|a|} \delta_{(\cos\theta + \frac{a}{k} + \frac{\Delta'}{ak} - \frac{\Omega'}{ak})} \qquad \text{(B.11)}
\end{aligned}
$$

with $\Omega' = \frac{m\Omega}{2\hbar}$ and $\Delta' = \frac{m\Delta}{2\hbar^2}$.

Because the on and off switching can be neglected, we only look at the linear term in Ω, which arises from the interval $k_F^+ + 2a > a + \frac{\Delta'}{a} - \frac{\Omega'}{a} > 2a - k_F^+$, where the plane intersects both Fermi spheres as indicated in figure B.5.

To derive k_{min}, we use the Phytagoraean theorem in the triangles again:

$$
\begin{aligned}
(k_F^-)^2 - \big(2a + (\frac{\Delta'}{a} - a + \frac{\Omega'}{a})\big)^2 &= (k_{min})^2 - (\frac{\Delta'}{a} - a + \frac{\Omega'}{a})^2 \\
\rightarrow (k_{min})^2 = (k_F^-)^2 - 4\Delta' - 4\Omega' &= (k_F^+)^2 - 4\Omega' \qquad \text{(B.12)}
\end{aligned}
$$

Therefore the result is exactly the same as in the other sum:

$$
\sum_k = \frac{V}{(2\pi)^2} \frac{m}{2\hbar^2 |a|} \int_{k_{min}}^{k_F^-} dk k = \frac{mV}{4\pi^2 \hbar^2 |a|} \Omega' \qquad \text{(B.13)}
$$

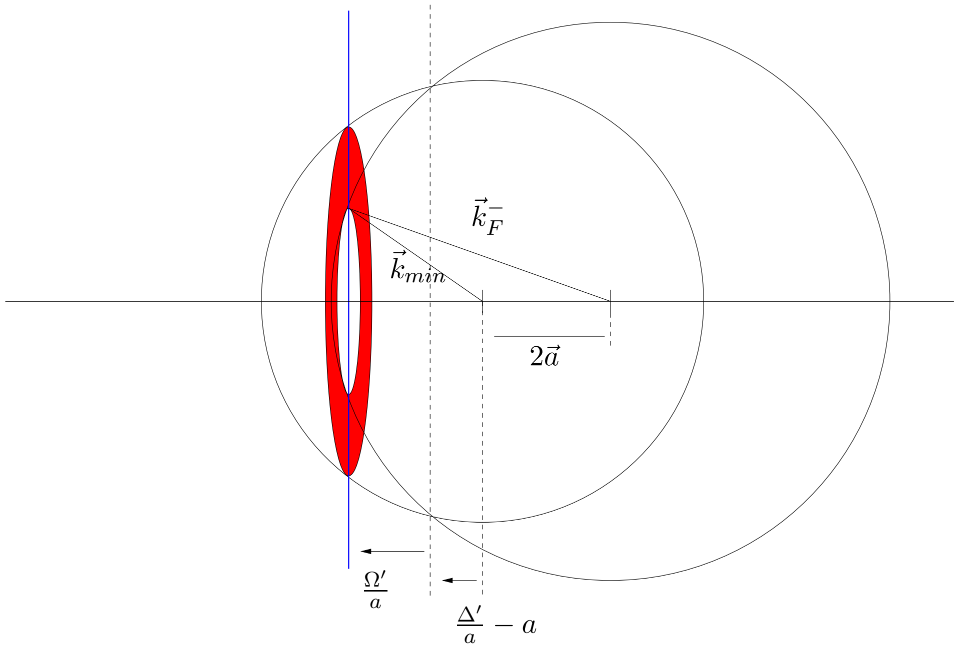


Figure B.5: Integration area for the second sum

Appendix C

Calculation of the Kernels with an applied Spin Current

C.1 $\partial_{\vec{q}} K_{Berry}\Big|_{q=0}$ the Gradient Term

In the current free case, the real part of the kernel has no dependence on $\vec{q}$ in the limit $\Omega = 0$. We proved this by using the property of the energy to be the same for $+\vec{k}$ and $-\vec{k}$. Now with the spin current this symmetry is broken.

In equation (2.43) we replace E with the altered particle energy $\widetilde{E}$, as defined in (5.8).

$$
\begin{aligned}
Re[K_{berry}]\Big|_{\Omega=0} &= \frac{U}{3N} P \sum_{\vec{k}} \frac{n_F(\widetilde{E}^+_{\vec{k}}) - n_F(\widetilde{E}^-_{\vec{k}+\vec{q}})}{\widetilde{E}^+_{\vec{k}} - \widetilde{E}^-_{\vec{k}+\vec{q}}} - \frac{n_F(\widetilde{E}^-_{\vec{k}}) - n_F(\widetilde{E}^+_{\vec{k}+\vec{q}})}{\widetilde{E}^-_{\vec{k}} - \widetilde{E}^+_{\vec{k}+\vec{q}}} \\
&= \frac{U}{3N} P \sum_{\vec{k}} n_F(\widetilde{E}^+_{\vec{k}}) \Big(\frac{1}{\widetilde{E}^+_{\vec{k}} - \widetilde{E}^-_{\vec{k}+\vec{q}}} + \frac{1}{\widetilde{E}^-_{\vec{k}-\vec{q}} - \widetilde{E}^+_{\vec{k}}} \Big) \\
&\qquad - n_F(\widetilde{E}^-_{\vec{k}}) \Big(\frac{1}{\widetilde{E}^+_{\vec{k}-\vec{q}} - \widetilde{E}^-_{\vec{k}}} + \frac{1}{\widetilde{E}^-_{\vec{k}} - \widetilde{E}^+_{\vec{k}+\vec{q}}} \Big) \qquad \text{(C.1)}
\end{aligned}
$$

We enter the quadratic dispersion relation, ignore the shift of the chemical potential and shift the summation to remove the $\vec{a}$ from the argument of the

Fermi distribution:

$$
\begin{aligned}
= \frac{U}{3N}\frac{m}{2\hbar^2}P\sum_{\vec{k}} \; & n_F(E^+_{\vec{k}})\Big(\frac{1}{\Delta' + ak\cos\theta - a^2 + aq\cos\chi - k\cos\eta\, q/2 - q^2/4} \\
& + \frac{1}{-\Delta' - ak\cos\theta + a^2 + aq\cos\chi - k\cos\eta\, q/2 + q^2/4}\Big) \\
& - n_F(E^+_{\vec{k}})\Big(\frac{1}{\Delta' + ak\cos\theta + a^2 - aq\cos\chi - k\cos\eta\, q/2 + q^2/4} \\
& + \frac{1}{-\Delta' - ak\cos\theta - a^2 - aq\cos\chi - k\cos\eta\, q/2 - q^2/4}\Big)
\end{aligned} \tag{C.2}
$$

With $\Delta' = \frac{m\Delta}{2\hbar^2}$ and the scalar products expressed by the angles. We always can Taylor expand in the spin wave momentum $|\vec{q}|$, because the momentum is neglectable to the typical electron momentum in a degenerate gas.

$$
\begin{aligned}
\approx \frac{U}{3N}\frac{m}{2\hbar^2}P\sum_{\vec{k}} \; & n_F(E^+_{\vec{k}})\Big(\frac{kq\cos\eta - 2aq\cos\chi}{(\Delta' - a^2 + ak\cos\theta)^2} \\
& - n_F(E^+_{\vec{k}})\Big(\frac{kq\cos\eta + 2aq\cos\chi}{(\Delta' + a^2 + ak\cos\theta)^2}
\end{aligned} \tag{C.3}
$$

We chose $\vec{a}$ as z-axes, so the angles θ and ϕ are our integration variables. Therefore χ, the angle between the direction of the current and the momentum of the spin wave, becomes a parameter, and η, the angle between $\vec{k}$ and $\vec{q}$ becomes a function of θ,ϕ and χ.

We can replace η by using the law of cosines in polar coordinates:

$$
\cos\eta = \cos\theta\cos\chi + sin\theta\sin\chi\cos\phi \tag{C.4}
$$

The term linear in $\cos\phi$ vanishes due to the angular integration, and we now

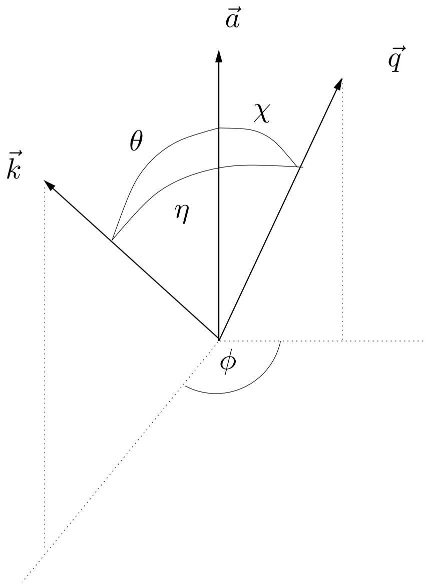


Figure C.1: Orientation of vectors for the integration over $\vec{k}$

have to calculate:

$$= \frac{U}{3N}\frac{m}{2\hbar^2} q\cos\chi\, P\sum_{\vec{k}} \quad n_F(E^+_{\vec{k}})\,\frac{k\cos\theta - 2a}{(\Delta' - a^2 + ak\cos\theta)^2}$$
$$-n_F(E^+_{\vec{k}})\,\frac{k\cos\theta + 2a}{(\Delta' + a^2 + ak\cos\theta)^2} \tag{C.5}$$

with standard integration techniques, equation (C.5) can be solved exact.

$$Re[K_{Berry}] = \frac{U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\, q\cos\chi\,\Big[\tag{C.6}$$
$$\frac{1}{2\,a^2}\big[\frac{3(\Delta' - a^2)^2}{a^2} - 4(a^2 - \Delta') - (k_F^+)^2\big]\,\log\Big|\frac{\Delta' - a^2 - a\,k_F^+}{\Delta' - a^2 + a\,k_F^+}\Big| + \frac{3\Delta k_F^+}{a^3} + \frac{k_F^+}{a}$$
$$-\frac{1}{2\,a^2}\big[\frac{3(\Delta' + a^2)^2}{a^2} - 4(\Delta' + a^2) - (k_F^-)^2\big]\,\log\Big|\frac{\Delta' + a^2 - a\,k_F^-}{\Delta' + a^2 + a\,k_F^-}\Big| - \frac{3\Delta k_F^-}{a^3} + \frac{k_F^-}{a}\Big]$$

To plot this complicated expression we scale the result by k_F, the Fermi wave vector of a system, with the same electron density, but without a Zeeman splitting. Therefore we replace $k_F^{\pm}/k_F = \sqrt{1 \mp 2\Delta'}$ and $\Delta'/k_F^2 = \Delta/(4\epsilon_F)$, so equation C.5 becomes

$$Re[K_\pi] = \frac{2U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2} \cdot X_q \tag{C.7}$$

with a numerical factor X_q, which is plotted in figure C.2 as solid line, together with its Taylor expansion (C.8) as dashed. ($\mu = 8eV, \Delta = 1eV$).

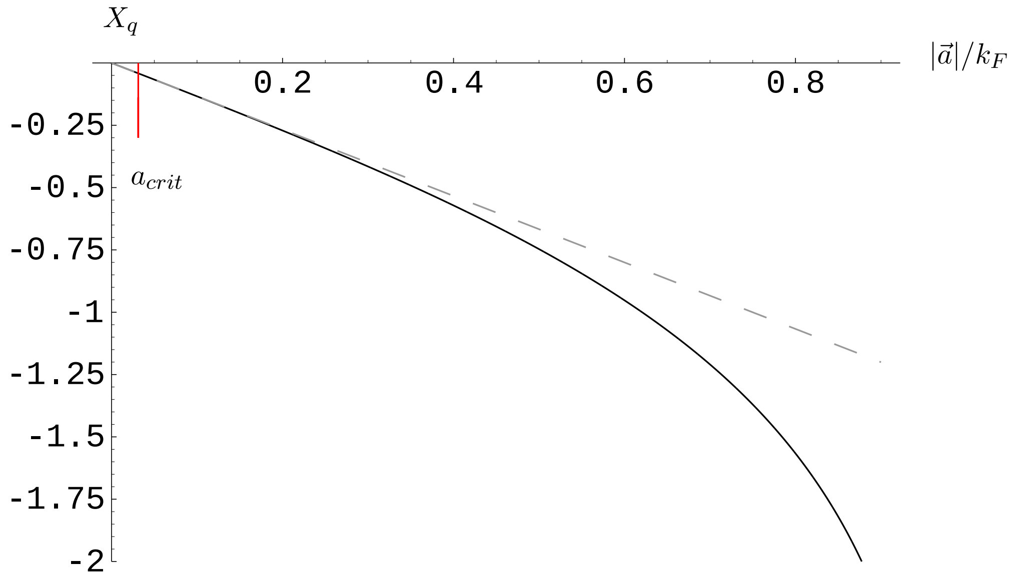


Figure C.2: Prefactor of $\vec{q}$

The marker a_{crit} is the current density, where the enhanced dissipation starts, which is discussed in the previous section. We see, the linear Taylor series is a very good approximation, even far beyond the critical current density. Therefore, we examine the Taylor series closer.

To avoid mathematical efforts, we Taylor expand equation C.2 in $|\vec{a}|$, before we integrate.

$$\approx \frac{2U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^3}\int_0^{2\pi}d\phi\int_{-1}^{1}d(cos\theta)\Big[\int_0^{k_F^+}k^2dk\big(-\frac{k^2\cos^2\theta}{\Delta'^3}-\frac{1}{\Delta'^2}\big) -\int_0^{k_F^-}k^2dk\big(-\frac{k^2\cos^2\theta}{\Delta'^3}+\frac{1}{\Delta'^2}\big)\Big]\vec{a}\vec{q} \qquad (C.8)$$

If we compare this expression with equation (2.56), we see, the integral is proportional to the spin wave stiffness ρ.

$$= -\frac{4\,\rho}{\Delta}\,\vec{a}\vec{q} \qquad (C.9)$$

C.2 $K_\pi\Big|_{q=0}$ the Mass Term

The diagonal kernel K_π generate a mass term, just like in the case of an applied external magnetic field. We take the $\Omega = 0$ and $\vec{q} = 0$ limit of (2.49).

$$Re[K_\pi] = 1 + \frac{2U}{3N}P\sum_{\vec{k}}\frac{n_F(\widetilde{E}^-_{\vec{k}}) - n_F(\widetilde{E}^+_{\vec{k}})}{\widetilde{E}^-_{\vec{k}} - \widetilde{E}^+_{\vec{k}}} \qquad (C.10)$$

The procedure is the same as in the previous Section. We insert the equation for $\widetilde{E}$, ignore the shift in the chemical potential, then shift the integration to remove $\vec{a}$ from the argument of the Fermi distribution.

$$\begin{aligned} Re[K_\pi] &= 1 + \frac{2U}{3N}P\sum_{\vec{k}}\frac{n_F(E^-_{\vec{k}-\vec{a}})}{-\Delta - 2\frac{\hbar^2\vec{a}\vec{k}}{m}} - \frac{n_F(E^+_{\vec{k}+\vec{a}})}{-\Delta - 2\frac{\hbar^2\vec{a}\vec{k}}{m}} \\ &= 1 + \frac{2U}{3N}P\sum_{\vec{k}}\frac{n_F(E^-_{\vec{k}})}{-\Delta - 2\frac{\hbar^2\vec{a}\vec{k}}{m} - 2\frac{\hbar^2a^2}{m}} - \frac{n_F(E^+_{\vec{k}})}{-\Delta - 2\frac{\hbar^2\vec{a}\vec{k}}{m} + 2\frac{\hbar^2a^2}{m}} \\ &= 1 + \frac{2U}{3N}\frac{m}{2\hbar^2}P\sum_{\vec{k}}\frac{-n_F(E^-_{\vec{k}})}{\Delta' + ak\cos\theta + a^2} + \frac{n_F(E^+_{\vec{k}})}{\Delta' + ak\cos\theta - a^2} \end{aligned} \qquad (C.11)$$

With the usual notation $\Delta' = \frac{m\Delta}{2\hbar^2}$ and θ as the angle between $\vec{a}$ and $\vec{k}$. By the use of this integral,

$$P\!\!\int_{-1}^{1}\! d(\cos\theta) \int_0^{k_F}\! dk \; \frac{k^2}{ak\cos\theta + b} \;\; = \;\; \frac{b\,k_F}{a^2} + \frac{1}{2\,a}\Big(\frac{b^2}{a^2} - k_F^2\Big) \,\log\Big|\frac{b - a\,k_F}{b + a\,k_F}\Big| \tag{C.12}$$

we can solve equation (C.11) exact

$$\begin{aligned} Re[K_\pi] &= \frac{2U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\Big[\qquad\qquad (C.13) \\ &\frac{2(k_F^-)^3}{3\Delta'} - \Big(\frac{(\Delta'+a^2)\,k_F^-}{a^2} + \frac{1}{2\,a}\Big(\frac{(\Delta'+a^2)^2}{a^2} - (k_F^-)^2\Big)\,\log\Big|\frac{\Delta'+a^2-a\,k_F^-}{\Delta'+a^2+a\,k_F^-}\Big|\Big) \\ -&\frac{2(k_F^+)^3}{3\Delta'} + \Big(\frac{(\Delta'-a^2)\,k_F^+}{a^2} + \frac{1}{2\,a}\Big(\frac{(\Delta'-a^2)^2}{a^2} - (k_F^+)^2\Big)\,\log\Big|\frac{\Delta'-a^2-a\,k_F^+}{\Delta'-a^2+a\,k_F^+}\Big|\Big)\Big] \end{aligned}$$

This calculation used that the magnetization and the direction of polarization is parallel. In the other case, the anti-parallel configuration, the vector potential gets an additional minus sign in equation 5.7. This is the same consequence as if we reverse the direction of the spin current. If we now look at equation C.13, the spin wave gap is an even function of the vector potential. Therefore, the spin current stabilizes the parallel and the anti-parallel configuration, antithetic to the case of the external magnetic field.

We again scale the result by k_F, just like equation C.7

$$Re[K_\pi] = \frac{2U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}k_F \,\cdot\, X_{gap} \tag{C.14}$$

with a numerical factor X_{gap}, which is plotted in figure C.3 as solid line, together with its quadratic Taylor expansion (C.15) as dashed.

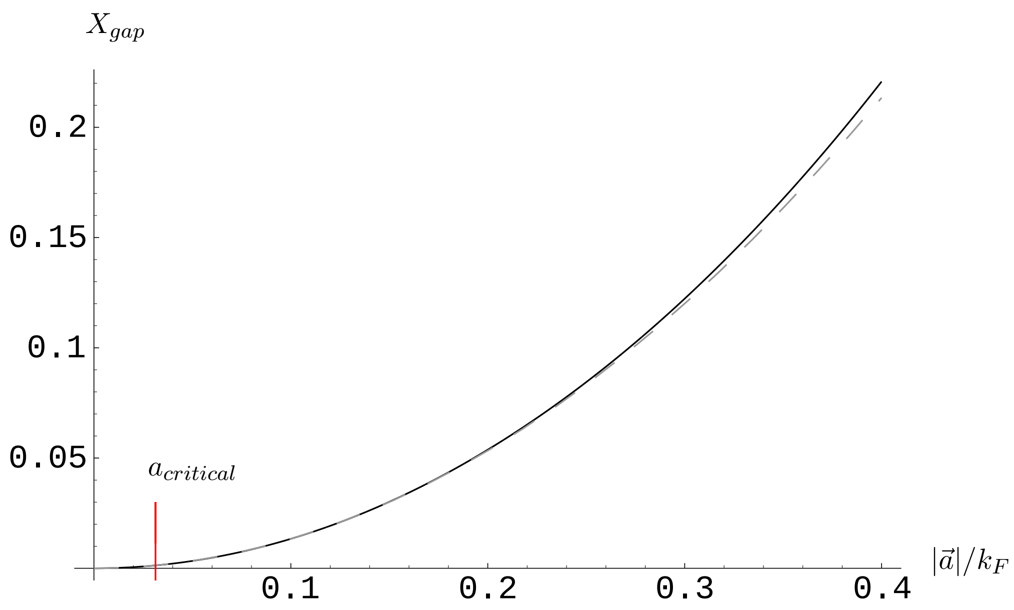


Figure C.3: Exact spin wave gap

As in the case of the gradient term, the Taylor expansion in $|\vec{a}|$ is a very good approximation for proper current densities. We Taylor expand equation C.11 before we integrate:

$$
\begin{aligned}
Re[K_\pi] &= \frac{2U}{3N} P \sum_{\vec{k}} \left[n_F(E^-_{\vec{k}}) + n_F(E^+_{\vec{k}}) \right] \frac{2\hbar^2 a^2}{m\,\Delta^2} \\
&\qquad + \left[n_F(E^+_{\vec{k}}) - n_F(E^-_{\vec{k}}) \right] \frac{4\hbar^4 k^2 a^2 \cos^2\theta}{m^2\,\Delta^3} \\
&= 4\frac{\rho}{\Delta}\vec{a}^2
\end{aligned}
\tag{C.15}
$$

C.3 $\partial^2_{\vec{q}} K_\pi \big|_{q=0}$ the former Spin Wave Stiffness

To calculate the spin wave dispersion without the weak current approximation, we need the spin wave stiffens (2.56) and the frequency prefactor

(2.45). The spin wave stiffness is the quadratic part of the the diagonal kernel.

$$
\begin{aligned}
Re[K_\pi]\Big|_{\Omega=0} = 4\frac{\rho}{\Delta}\vec{a}^2 + \frac{U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\, q^2 \Big[\Big(&\frac{(k_F^-)^2(1+3\cos[2\chi])}{16a^3} \\
-\frac{a^4+2a^2\Delta'+9\Delta'^2-(a^4-6a^2\Delta'-15\Delta'^2)\cos[2\chi]}{16a^5}\Big)\log\Big|&\frac{\Delta'+a^2-ak_F^-}{\Delta'+a^2+ak_F^-}\Big| \\
-\frac{(-a^2+7\Delta'-(3a^2-13\Delta')\cos[2\chi])k_F^-}{8a^4} - &\frac{(a^2-\Delta')^2(a^2+\Delta')\cos^2[\chi]\,k_F^-}{2a^4(a^2+\Delta'-ak_F^-)(a^2+\Delta'+ak_F^-)} \\
\\
+\Big(\frac{(k_F^+)^2(1+3\cos[2\chi])}{16a^3} - &\frac{a^4-2a^2\Delta'+9\Delta'^2-(a^4+6a^2\Delta'-15\Delta'^2)\cos[2\chi]}{16a^5}\Big)\cdot \\
\cdot\log\Big|\frac{\Delta'-a^2+ak_F^+}{\Delta'-a^2-ak_F^+}\Big| & \\
-\frac{(-a^2-7\Delta'-(3a^2+13\Delta')\cos[2\chi])k_F^+}{8a^4} - &\frac{(a^2-\Delta')(a^2+\Delta')^2\cos^2[\chi]\,k_F^+}{2a^4((a^2-\Delta')^2-a^2k_F^{+2})}\Big]
\end{aligned}
\tag{C.16}
$$

We plot the result in figure C.4 with the same conventions as before. Because this term depend on the angle between current and spin wave momentum, we plot the parallel and perpendicular case. As we can see on the X-axis, the variation of the spin wave stiffness is quite small, and the Taylor expansion zero order in a is a good approximation up to the critical current density.

So the exact result will just like equation C.16 not produce an instability.

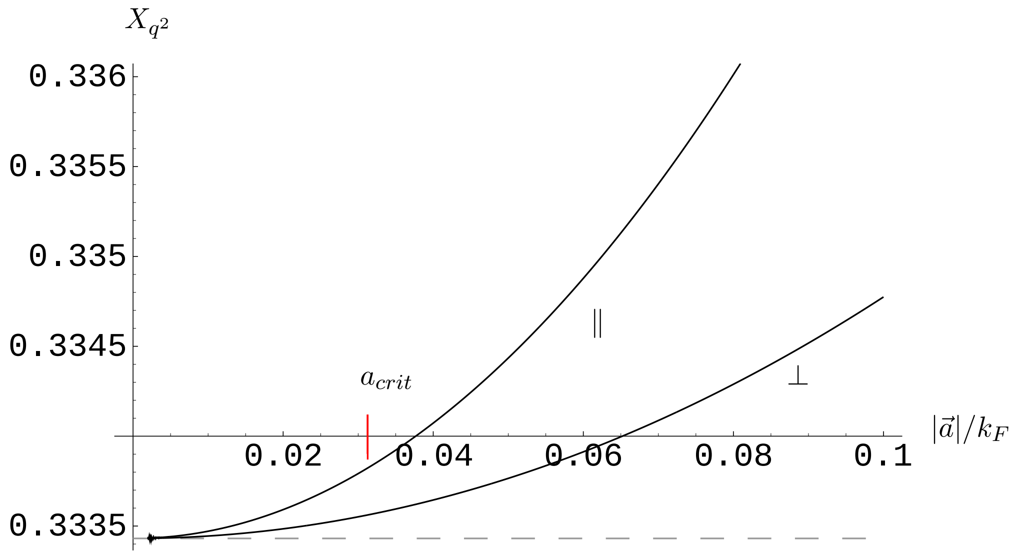


Figure C.4: Spin wave stiffness as a function of the current

C.4 $\partial_\Omega K_{Berry}\big|_{q=0}$ the Energy Proportionality Factor

For completeness, we examine the Ω dependence of the Berry phase kernel. Therefore, we insert the particle energy 5.8 in the equation for $K_{berry}\big|_{\vec{q}=0}$ (2.45). the result is

$$Re[K_{Berry}]\Big|_{\vec{q}=0} = \frac{U}{3N}\frac{m}{2\hbar^2}\frac{V}{(2\pi)^2}\,\Omega\,\Big[\,\frac{2a(k_F^+ - k_F^-)}{a^3}$$
$$-\frac{a^2-\Delta'}{a^3}\log\Big|\frac{\Delta' - a^2 - ak_F^+}{\Delta' - a^2 + ak_F^+}\Big| - \frac{a^2+\Delta'}{a^3}\log\Big|\frac{\Delta' + a^2 - ak_F^-}{\Delta' + a^2 + ak_F^-}\Big| \qquad \text{(C.17)}$$

We print it in the usual manner in figure C.5 The prefactor diverges at the critical current density, so there the energy of creating magnons with arbitrary momentum vanishes.

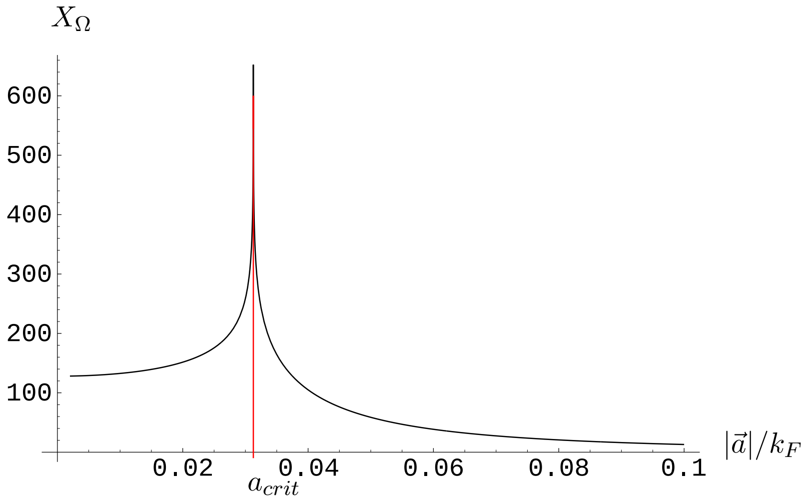


Figure C.5: Proportionality factor of Ω

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Vita

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